COMPLEX HYPERBOLIC STRUCTURES ON DISC BUNDLES OVER SURFACES

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ABSTRACT. We study complex hyperbolic disc bundles over closed orientable surfaces that arise from discrete and faithful representations $H_n \to \mathrm{PU}(2,1)$, where H_n is the fundamental group of the orbifold $\mathbb{S}^2(2, \dots, 2)$ and thus contains a surface group as a subgroup of index 2 or 4. The results obtained provide the first complex hyperbolic disc bundles $M \to \Sigma$ that
- admit both real and complex hyperbolic structures,
- satisfy the equality $2(\chi + e) = 3\tau$,
- satisfy the inequality $\frac{1}{2}\chi < e$, and
- induce discrete and faithful representations $\pi_1\Sigma \to \mathrm{PU}(2,1)$ with fractional Toledo invariant,

where χ is the Euler characteristic of Σ, e denotes the Euler number of M, and τ stands for the Toledo invariant of M. To get a satisfactory explanation of the equality $2(\chi + e) = 3\tau$, we conjecture that there exists a holomorphic section in all our examples.

In order to reduce the amount of calculations, we systematically explore coordinate-free methods.

CONTENTS

1. Introduction

This paper is devoted to a particular case of the traditional question concerning the interrelation between topology and geometry: Does a topological 4-manifold admit a specific geometric structure (real hyperbolic, complex hyperbolic, quaternionic, etc.)? Henceforth, M denotes an oriented disc bundle over a closed orientable surface Σ of Euler characteristic χ and e stands for the Euler number of the bundle.

For the existence of real hyperbolic structure on M, there are various conditions in terms of χ and e (see [GLT], [Kui], [Luo], and also [Kap1, Kap2]). We study the case of complex hyperbolic geometry. In this case, there is one more discrete invariant, the Toledo invariant of the representation $\pi_1\Sigma \to \mathrm{PU}(2,1)$ provided by M. It is related to the complex (= Riemannian) structure on M, takes

values in $\frac{2}{3}\mathbb{Z}$, and satisfies the inequality $|\tau| \leq |\chi|$ [Tol]. The Toledo invariant is the only discrete invariant of a (not necessarily discrete) representation $\pi_1\Sigma \to \mathrm{PU}(2,1)$ [Xia].

There are not so many known complex hyperbolic disc bundles. Simple types of such bundles are the $\mathbb{R}$-Fuchsian bundles (they satisfy $e = \chi$ and $\tau = 0$) and the $\mathbb{C}$-Fuchsian bundles (they are characterized by $\chi = \tau$ [Tol] and satisfy $e = \frac{1}{2}\chi$). The first complex hyperbolic bundles different from $\mathbb{R}$- and $\mathbb{C}$-Fuchsian ones were constructed in [GKL]. They satisfy the relations $e = \chi + |\tau/2|$ and $\chi \leq e \leq \frac{1}{2}\chi$. Thus, $\mathbb{R}$- and $\mathbb{C}$-Fuchsian bundles provide the extreme values of e in all examples known before the present paper.

Here we consider a class of discrete and faithful representations that is slightly simpler than that of general representations $\pi_1\Sigma \to \mathrm{PU}(2,1)$. Let H_n denote the group generated by $r_1, \ldots, r_n$ with the defining relations $r_n \ldots r_1 = 1$ and $r_i^2 = 1$, $i = 1, \ldots, n$. For a hyperelliptic Σ, the group H_n is nothing but the index two extension of $\pi_1\Sigma$ by the hyperelliptic involution of Σ and n is even (for odd n, a surface group is a subgroup of index 4 in H_n). We study discrete and faithful representations $\varrho : H_n \to \mathrm{PU}(2,1)$ such that the ϱr_i's are reflections in ultraparallel complex geodesics. In order to prove discreteness, we construct fundamental polyhedra bounded by cycles of segments of bisectors (see Outline at the end of Introduction for more details). The main difficulties at these stages are to check that segments of bisectors intersect properly and the fact that adjacent segments have nonconstant angle along their intersection.

The oriented transversal triangles of bisectors studied in Subsections 2.5 and 4.4 serve as building blocks for fundamental polyhedra. The vertices of such a triangle are complex geodesics (= Poincaré discs) naturally equipped with an isometry called the holonomy of the triangle. Every oriented transversal triangle of bisectors bounds a (closed) fibred polyhedron, i.e., a disc bundle over a disc that extends the slice bundle structure of the triangle itself. This fact is established by showing that the space of oriented transversal triangles of bisectors is path-connected and that a triangle of bisectors with common complex spine bounds a fibred polyhedron. It is crucial that the holonomy of a triangle cannot be R-parabolic (see 2.5.1 for the definition) nor the identity. Moreover, every oriented transversal triangle of bisectors possesses a 'fractional Euler quantity' that can actually be interpreted as its Euler number. The fractional Euler quantity is an arc on the ideal boundary of a vertex of the triangle. It is determined by the holonomy of the triangle and is additive with respect to gluing triangles.

Gluing a couple of transversal triangles of bisectors, we obtain in Section 3 a quadrangle of bisectors bounding a fundamental polyhedron for the group generated by $U, W \in \mathrm{PU}(2,1)$ with the relations $U^n = W^n = (U^{-1}W)^2 = 1$. The quadrangle furnishes a complex hyperbolic disc bundle over the turnover orbifold $\mathbb{S}^2(n, n, 2)$ whose rational Euler number (see [Fen] for the definition) can be inferred from the fractional Euler quantities of the involved triangles. Since H_n is a subgroup of index n in the turnover group, we get a complex hyperbolic disc bundle M over a closed surface. Clearly, we simultaneously obtain a compact 3-manifold (a circle bundle over a closed surface) admitting spherical CR-structure.

In the above way, we construct a huge family of explicit examples of disc bundles M. All of them satisfy the inequality $\frac{1}{2}\chi < e$ and the equality $2(\chi + e) = 3\tau$ (with negative χ, e, τ). The inequality was never valid for previously known examples and the equality was valid only in the $\mathbb{C}$-Fuchsian case. Since the equality is a necessary condition for the existence of a holomorphic section of M, we conjecture that there exists a holomorphic section in all our examples. For many examples, τ is not integer, which implies in particular that the corresponding representation $\pi_1\Sigma \to \mathrm{PU}(2,1)$ cannot be lifted to $\mathrm{SU}(2,1)$. We obtain the first disc bundles M admitting both real and complex hyperbolic structures. Passing to the corresponding circle bundles, we see that there exist circle bundles over closed orientable surfaces admitting simultaneously conformally flat and spherical CR-structures. (See also the beautiful examples constructed by Schwartz in [Sch1, Sch2].) For a more detailed discussion about the features of our examples and related conjectures, see Subsection 3.3.

Whenever reasonable, we work without coordinates. This concerns especially the appendix (Section 4)

which is devoted to an exposition of algebraic and geometric backgrounds and contains the proofs absent in Sections 2–3. We believe that some facts in the appendix are interesting *per se*. This includes: the explicit formulae for the normal vector to an oriented bisector (Proposition 4.2.11) and for the angle between two cotranchal bisectors (Lemma 4.3.1); the numerical criterion of transversality of two cotranchal bisectors (Criterion 4.3.3); a sort of metric separability of transversal cotranchal bisectors (Lemma 4.3.6) that is a key point allowing to apply Poincaré's Polyhedron Theorem; and the explicit formula for a Kähler potential in 4.5.1.

The reader unfamiliar with complex hyperbolic geometry may begin reading with Section 4. The reader familiar with complex hyperbolic geometry can read first Sections 2–3 (the geometric core of the paper).

Outline of general construction. Roughly speaking, the general construction of a complex hyperbolic disc bundle M deals with a certain fundamental polyhedron F for the group $\pi_1 M \subset \mathrm{PU}(2,1)$. The polyhedron F is bounded by a cycle of oriented segments of bisectors $(\mathrm{B}_1, \ldots, \mathrm{B}_m)$ such that the final slice of B_i is the initial slice of B_{i+1} for every i (the indices are modulo m). Suppose that the obvious slice bundle structure of the boundary $\partial_0 F := \mathrm{B}_1 \cup \cdots \cup \mathrm{B}_m$ of F is extendable to F. Then, taking into account that the identifications of the segments by means of elements in $\pi_1 M$ preserve the slice bundle structure of $\partial_0 F$, we obtain a complex hyperbolic disc bundle M.

For technical reasons, the polyhedron F is glued from two or four polyhedra P, each bounded by a cycle $\mathcal{C}$ of segments $(\mathrm{B}_1, \ldots, \mathrm{B}_n)$. The identification of faces of P is given by the reflections R_i in the middle slices M_i of the B_i's which satisfy the relation $R_n \ldots R_1 = 1$. If we show that P is a fundamental polyhedron for the group H generated by the R_i's and that the slice bundle structure of $\partial_0 P$ is extendable to P, we arrive at the desired complex hyperbolic disc bundle M.

The solid torus $\partial_0 P$ can be readily fibred into circles called meridional curves of $\mathcal{C}$: in its segment b_i belonging to B_i, every meridional curve b is equidistant to the real spine of B_i and lies in a suitable meridian of B_i. Moreover, $R_i b_i = b_i$ and R_i acts on b_i as a 'reflection' in the middle point of b_i, i.e., in the point $b_i \cap M_i$. In other words, the reflections R_i's preserve the fibering of $\partial_0 P$ by meridional curves. Every meridional curve of the cycle $\mathcal{C}$ is intended to generate a section of the disc bundle, i.e., to be the boundary of a disc that provides a section.

Considering those 'formally' adjacent polyhedra conjugate to P that are intended to tessellate a neighbourhood of the slice S_{i+1} common to the segments B_i and B_{i+1}, we can see that the sum of the interior angles of these adjacent polyhedra at a point $s \in S_{i+1}$ (called the total angle at s) is an integer multiple of 2π and that the total angle at s is independent of the choice of s in S_{i+1}. Requiring that the total angle is 2π and that the full bisectors $\prec \mathrm{B}_i \succ$ and $\prec \mathrm{B}_{i+1} \succ$ of the segments B_i and B_{i+1} are transversal along their common slice S_{i+1} for all i (such a cycle $\mathcal{C}$ is said to be simple transversal with total angle 2π), we show that P is indeed a fundamental polyhedron for H.

It is not so easy to construct a simple transversal cycle of bisectors. The problem is that the segments B_i and B_j can unexpectedly intersect, that is, the cycle $\mathcal{C}$ can be nonsimple and, thus, can provide no polyhedron P. Under some additional requirement (appealing to some kind of simple Poincaré duality), we show that the cycle is simple.

In general, we cannot expect the slice bundle structure of $\partial_0 P$ to be extendable to P. It is quite possible that the polyhedron P is 'knotted'. Although this is a topological problem, it looks even more difficult than the problem of simplicity because it requires to dwell into the mutual position of all the B_i's (not only to verify that the B_i's intersect properly). An easy way to avoid such a situation is to show that the polyhedron P can be cut into some simple polyhedra fibred in a way compatible to gluing them back. Besides, this helps to solve the problem of simplicity of a cycle. Transversal triangles of bisectors are good candidates for such simple fibred polyhedra.

important suggestions that considerably improved the exposition. We thank Israel Vainsencher for a valuable remark. This article is dedicated to the memory of Igor′ Vladimirovich L′vov.

<div style="text-align:center">

Some notation

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2. General construction

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In this section, we develop some general tools applicable to constructing complex hyperbolic disc bundles over closed orientable surfaces.

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2.1. Preliminaries. Cycle of bisectors and Toledo invariant

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In this subsection, we introduce some basic notation, conventions, and simple facts. Then we define configuration and cycle of bisectors, meridional curves, and related concepts. Finally, we calculate the Toledo invariant of representations provided by a cycle of bisectors.

2.1.1. Fix, once and for all, a three-dimensional $\mathbb{C}$-vector space V equipped with an hermitian form $\langle -, - \rangle$ of signature $+ + -$. The corresponding unitary, special unitary, and projective unitary groups are respectively denoted by U, SU, and PU. Depending on the context, we will use the elements of V to denote the points in the complex projectivization $\mathbb{P}$ of V. In general, $\mathbb{P}W$ denotes the complex projectivization of $W \subset V$.

It is well known that the complex hyperbolic plane can be identified with the open 4-ball[1]

$$\mathbb{B} := \big\{ p \in \mathbb{P} \mid \langle p, p \rangle < 0 \big\}.$$

The complex hyperbolic distance between two points $p_1, p_2 \in \mathbb{B}$ can be expressed in terms of the *tance*

$$\mathrm{ta}(p_1, p_2) := \frac{\langle p_1, p_2 \rangle \langle p_2, p_1 \rangle}{\langle p_1, p_1 \rangle \langle p_2, p_2 \rangle} \tag{2.1.2}$$

as explained in [Gol, p. 77]. Sometimes, it is convenient to use the tance instead of the distance: the latter is a monotonic function of the first, whereas the tance has an algebraic nature and a simpler form. Furthermore, the tance is involved in many complex hyperbolic concepts.

The ideal boundary of the complex hyperbolic plane is the 3-sphere

$$\partial_\infty \mathbb{B} := \big\{ p \in \mathbb{P} \mid \langle p, p \rangle = 0 \big\}$$

formed by all isotropic points. We denote $\overline{\mathbb{B}} := \mathbb{B} \cup \partial_\infty \mathbb{B}$. Preferentially, we consider geometrical objects such as geodesics, complex geodesics, $\mathbb{R}$-planes, bisectors, etc. as being closed in $\overline{\mathbb{B}}$. The exceptions occur when explicitly stated or when a concept or formula is clearly inapplicable to isotropic points. In general, $\partial_\infty X$ stands for $\partial_\infty \mathbb{B} \cap X$.

For every nonisotropic $p \in \mathbb{P}$, define the linear map

$$R(p) : x \mapsto 2 \frac{\langle x, p \rangle}{\langle p, p \rangle} p - x. \tag{2.1.3}$$

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[1]Here and in what follows, the symbol := stands for 'equals by definition.'

Such a definition provides the reflection $R(p) \in \mathrm{SU}$. If p is positive, that is, if $p \notin \overline{\mathbb{B}}$, then $R(p)$ is the reflection in the complex geodesic $\mathbb{P}p^\perp \cap \overline{\mathbb{B}}$, where $p^\perp := \{x \in V \mid \langle x, p \rangle = 0\}$. The point p is said to be the *polar point* to $\mathbb{P}p^\perp$. For brevity, $\mathbb{P}p^\perp$ will denote both the complex geodesic $\mathbb{P}p^\perp \cap \overline{\mathbb{B}}$ and its projective line, depending on the context.

It is well known that every bisector in $\overline{\mathbb{B}}$ admits two decompositions into totally geodesic 2-planes: the slice decomposition into complex geodesics and the meridional decomposition into $\mathbb{R}$-planes (see [Gol, p. 152]). We treat every bisector as being oriented with respect to the orientation of its real spine and the natural (complex) orientation of its slices. Let S and S' be two complex geodesics. The inequality $\mathrm{ta}(p, p') > 1$ means that S and S' are ultraparallel [Gol, p. 100], where p and p' stand for the polar points to S and S'. In this case, there exists a unique bisector $\mathcal{B}$ containing S and S' as slices [Gol, p. 165, Theorem 5.2.4]. We denote by $\mathrm{B}[S, S'] \subset \mathcal{B}$ the closed oriented *segment of bisector* formed by the slices from S to S' and call $\mathcal{B}$ the *full bisector* of the segment $\mathrm{B}[S, S']$. It is easy to see that there exists a unique slice M of $\mathrm{B}[S, S']$ such that the reflection in M exchanges S and S'. We call M the *middle slice* of $\mathrm{B}[S, S']$.

Let $\mathrm{B}[S, S']$ be an oriented segment of bisector and let $s \in S$. We take a meridian R of $\mathrm{B}[S, S']$ such that $s \in \mathrm{R}$. Let b denote the curve contained in R that begins with $s \in S$, ends with some point $s' \in S'$, and is equidistant to the real spine of the bisector (b is a segment of hypercycle in R). We call b the *meridional curve* of $\mathrm{B}[S, S']$ generated by $s \in S$. When $s \in \partial_\infty S$, the *ideal meridional curve* b is defined as the segment of $\mathbb{R}$-circle in $\mathrm{B}[S, S']$ that contains s.

Lemma 2.1.4. *Let* $\mathrm{B}[S, S']$ *be an oriented segment of bisector and let* b *be an* (ideal) *meridional curve of* $\mathrm{B}[S, S']$ *that begins with* $s \in S$ *and ends with* $s' \in S'$. *Then* $Rb = b$ *and* $Rs = s'$, *where* R *stands for the reflection in the middle slice of* $\mathrm{B}[S, S']$.

Proof. There is a meridian R of $\mathrm{B}[S, S']$ such that $b \subset \mathrm{R}$. The reflection in the middle slice M of $\mathrm{B}[S, S']$, being restricted to R, is the reflection in the geodesic $\mathrm{R} \cap M$ that is orthogonal to the real spine of $\mathrm{B}[S, S']$. ∎

Lemma 2.1.5 [Appendix, Lemma 4.2.16]**.** *An* (ideal) *meridional curve depends continuously on the segment of bisector and on the initial point of the curve.*

2.1.6. Let $S_1, \dots, S_n$ be complex geodesics such that S_i and S_{i+1} are ultraparallel for all i (the indices are modulo n). Denote $\mathrm{B}_i := \mathrm{B}[S_i, S_{i+1}]$. We call $(\mathrm{B}_1, \dots, \mathrm{B}_n)$ a *configuration of bisectors*. For every B_i, we denote by M_i the middle slice of B_i and by R_i, the reflection in M_i.

A configuration of bisectors $(\mathrm{B}_1, \dots, \mathrm{B}_n)$ is called *simple* if $\mathrm{B}_i \cap \mathrm{B}_j \neq \varnothing$ implies that $j \in \{i-1, i, i+1\}$ and if $\mathrm{B}_{i-1} \cap \mathrm{B}_i = S_i$ for all i. A configuration is *transversal* if, for all i, $\mathrm{B}_{i-1} \cap \mathrm{B}_i = S_i$ and the full bisectors of B_{i-1} and B_i are transversal along S_i (this includes transversality along $\partial_\infty S_i$).

Definition 2.1.7. A *cycle of bisectors* is a configuration of bisectors $(\mathrm{B}_1, \dots, \mathrm{B}_n)$ such that the reflections R_i's in the middle slices M_i's satisfy the relation $R_n \dots R_1 = 1$ in PU. In the terms of SU, identifying the centre of U with the unit complex numbers, we have $R_n \dots R_1 = \delta$, where $\delta \in \mathbb{C}$ and $\delta^3 = 1$. We denote by $H := \langle R_i \mid 1 \le i \le n \rangle$ the subgroup in PU generated by the R_i's.

Given reflections $R_1, \dots, R_n$ in complex geodesics $M_1, \dots, M_n$ subject to the relation $R_n \dots R_1 = 1$ in PU, a cycle of bisectors can be constructed as follows. Let S_1 be a complex geodesic. Define $S_{i+1} := R_i S_i$. Requiring that M_i and S_i are ultraparallel for all i, we obtain the segments $\mathrm{B}_i := \mathrm{B}[S_i, S_{i+1}]$ forming the cycle $(\mathrm{B}_1, \dots, \mathrm{B}_n)$. Constructing a cycle in this way does depend on the choice of S_1. There are many suitable choices for S_1. Indeed, for some positive points m_i and p_1, we have $R_i = R(m_i)$, $M_i = \mathbb{P}m_i^\perp$, $S_1 = \mathbb{P}p_1^\perp$, and $S_i := \mathbb{P}p_i^\perp$, where $p_{i+1} := R_i p_i$. In these terms, the inequality $\mathrm{ta}(m_i, p_i) > 1$ says that M_i and S_i are ultraparallel. We have $\mathrm{ta}(m_i, p_i) = \mathrm{ta}(R_1 \dots R_{i-2} R_{i-1} m_i, p_1)$. All these tances tend to infinity while $p_1 \notin \overline{\mathbb{B}}$ tends to a generic point in $\partial_\infty \mathbb{B}$.

2.1.8. Meridional curve of a cycle. Let $(B_1, \ldots, B_n)$ be a configuration of bisectors. Take $s_1 \in S_1$. The point s_1 generates a meridional curve b_1 of B_1 which ends with some $s_2 \in S_2$. By Lemma 2.1.4, $R_1 b_1 = b_1$ and R_1 exchanges s_1 and s_2. Inductively, s_i generates the meridional curve b_i of B_i which ends with some $s_{i+1} \in S_{i+1}$. Again, $R_i b_i = b_i$, $R_i s_i = s_{i+1}$, and $R_i s_{i+1} = s_i$. We call[2] the curve $b := b_1 \cup \cdots \cup b_n$ the *meridional curve* of the configuration generated by s_1. Similarly, $s_1 \in \partial_\infty S_1$ generates an *ideal meridional curve*. The arcs $b_1, \ldots, b_n$ are *edges* and the points $s_1, \ldots, s_n$ are *vertices* of the (ideal) meridional curve. In the case of a cycle, the meridional curves are closed due to the relation $R_n \ldots R_1 = 1$, as illustrated in the picture.
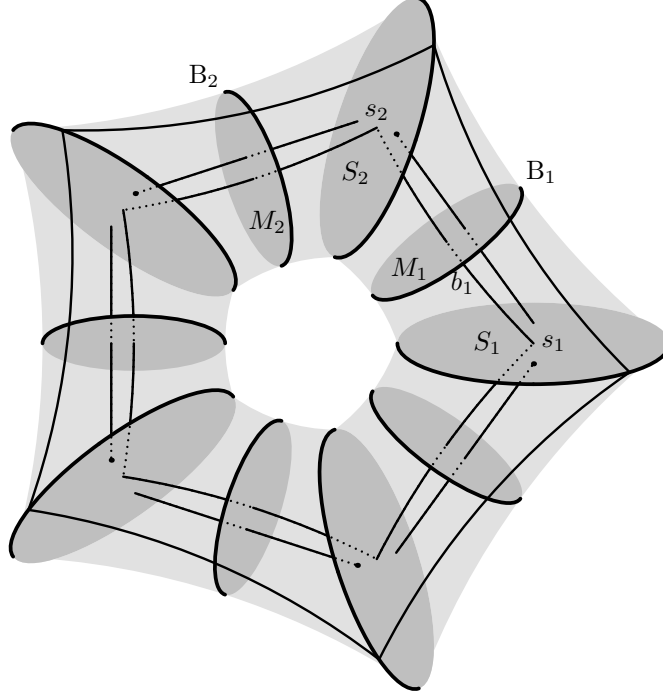
2.1.9. Plane example. The concept of cycle of bisectors in $\mathbb{B}$ originates from the following plane example.

Fix an integer $n \geq 5$. Let P be a simply connected geodesic n-gon in the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$ with vertices $v_1, \ldots, v_n$ and interior angles $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1 + \cdots + \alpha_n = 2\pi$. We denote the edges of P by $e_1, \ldots, e_n$, where e_i connects v_i and v_{i+1} (the indices are modulo n). Let r_i stand for the reflection in the middle point of e_i. By Poincaré's Polyhedron Theorem, P provides a fundamental domain for the group

$$H_n := \langle r_1, \ldots, r_n \mid r_n \ldots r_1 = 1 \text{ and } r_i^2 = 1 \rangle \qquad (\textbf{2.1.10})$$

generated by the r_i's subject to the defining relations $r_n \ldots r_1 = 1$ and $r_i^2 = 1$. The group H_n is the fundamental group of the sphere with n elliptic points of order 2, i.e., of the orbifold $\mathbb{S}^2(2, \ldots, 2)$. Passing to a torsion-free subgroup of finite index, we arrive at surface groups as follows.

For even n, $P \cup r_1 P$ provides a fundamental region for the subgroup

$$G_n := \langle r_1 r_i \mid 2 \leq i \leq n \rangle$$

of index 2, implying that G_n is the fundamental group of a closed orientable surface of genus $\frac{n}{2} - 1$ (the polygon $P \cup r_1 P$ has two cycles of vertices and $n - 1$ pairs of edges to identify).

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[2]For two curves c_1 and c_2 such that c_2 begins with the final point of c_1, we denote by $c_1 \cup c_2$ their concatenation.

For odd n, the polygon $P \cup r_1 P \cup r_2 P \cup r_2 r_1 P$ provides a fundamental region for the group T_n of index 4 generated by

$$x := r_2 r_1 r_n, \qquad y := r_2 r_n r_1, \qquad z := r_2 r_1 r_2 r_1, \qquad x_i := r_1 r_i, \qquad y_i := r_2 r_1 r_i r_2,$$

where $3 \leq i \leq n-1$,
$$T_n := \langle x, y, z, x_i, y_i \mid 3 \leq i \leq n-1 \rangle,$$

implying that T_n is the fundamental group of a closed orientable surface of genus $n-3$.

2.1.11. Toledo invariant. Following [Tol], one can introduce the Toledo invariant of a representation in PU of an arbitrary group with discrete and cocompact action on a plane (see also [Kre]). Every cycle of bisectors $(B_1, \ldots, B_n)$ provides a representation of H_n in PU given by $r_i \mapsto R_i$. In order to calculate the Toledo invariant of this representation, we express a Kähler potential in coordinate-free terms. In fact, such a coordinate-free approach is kind of a motto in this article. It makes possible to significantly simplify the complexity of formulae.

We can regard every linear map $\varphi \in \mathrm{Lin}_\mathbb{C}(\mathbb{C}p, V)$ as a tangent vector $t_\varphi \in \mathrm{T}_p \mathbb{P}$ by defining

$$t_\varphi f := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\big(p + \varepsilon\varphi(p)\big)$$

for a local smooth function f on $\mathbb{P}$ and its lift $\hat{f}$ to V. In this known way, we identify the tangent space $\mathrm{T}_p \mathbb{P}$ with the $\mathbb{C}$-vector space $\mathrm{Lin}_\mathbb{C}(\mathbb{C}p, V/\mathbb{C}p)$ of linear maps. In particular, for every $p \notin \partial_\infty \mathbb{B}$,

$$\mathrm{T}_p \mathbb{P} \simeq \langle -, p \rangle p^\perp.$$

(Of course, $\langle -, p \rangle p^\perp$ means $\langle -, p \rangle \otimes_\mathbb{C} p^\perp$.) For a nonisotropic $p \in V$ and for $v \in p^\perp$, we denote

$$v_p := \langle -, p \rangle v \in \mathrm{T}_p \mathbb{P}.$$

Thus, $\mathrm{T}_p \mathbb{P}$ is equipped with the hermitian form defined by $\langle v_p, w_p \rangle := -\langle p, p \rangle \langle v, w \rangle$, where $v, w \in p^\perp$. We arrived at a positive definite *hermitian metric* on the complex hyperbolic plane $\mathbb{B}$ that coincides, up to the scale factor of 4, with the one introduced in [Gol, p. 77] (see Corollary 4.1.18).

Lemma 2.1.12 [Appendix, Propositions 4.5.6 and 4.5.2]**.** *Let* $0 \neq c \in V$. *Define the 1-form* P_c *by the rule*
$$P_c(v_p) := -\mathrm{Im}\left(\frac{\langle p, p \rangle \langle v, c \rangle}{2 \langle p, c \rangle}\right)$$

for all $p \notin \mathbb{P}c^\perp \cup \partial_\infty \mathbb{B}$. *Let* $c_1, c_2 \in V$, $\langle c_1, c_2 \rangle \neq 0$. *Define the function* f_{c_1, c_2} *by the rule*[3]

$$f_{c_1, c_2}(p) := \frac{1}{2} \mathrm{Arg}\left(\frac{\langle c_1, p \rangle \langle p, c_2 \rangle}{\langle c_1, c_2 \rangle}\right)$$

for all $p \notin \mathbb{P}c_1^\perp \cup \mathbb{P}c_2^\perp$. *Then* $dP_c = \omega$ *and* $P_{c_1} - P_{c_2} = df_{c_1, c_2}$, *where, for* $p \in \mathbb{B}$ *and* $v_p, w_p \in \mathrm{T}_p \mathbb{B}$, *the Kähler form on* $\mathbb{B}$ *is given by* $\omega(v_p, w_p) := \mathrm{Im}\langle v_p, w_p \rangle$.

Lemma 2.1.13 [Appendix, Corollary 4.2.3]**.** *Let* R *be an* $\mathbb{R}$-*plane and let* $p, c \in$ R. *Then* $P_c(v_p) = 0$ *for every* $v_p \in \mathrm{T}_p$ R.

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[3]In this lemma, it is better to read Arg as a multi-valued and, hence, smooth function defined in $\mathbb{C} \setminus \{0\}$.

Lemma 2.1.14 [Appendix, Lemma 4.1.16]**.** *If* $p, g_1, g_2 \in \mathbb{B}$, *then* $\dfrac{\langle g_1, p\rangle\langle p, g_2\rangle}{\langle g_1, g_2\rangle}$ *cannot be real nonnegative.*

We are able now to calculate the Toledo invariant.

Proposition 2.1.15. *Let* $(\mathrm{B}_1, \ldots, \mathrm{B}_n)$ *be a cycle of bisectors and let* $\varrho : H_n \to \mathrm{PU}$ *be the representation given by* $r_i \mapsto R_i$. *Then the Toledo invariant of* ϱ *satisfies* $\tau \equiv n - \dfrac{\mathrm{Arg}\,\delta}{\pi}$ mod 2, *where* δ *is introduced in Definition 2.1.7.*

Proof. We have $R_i = R(m_i)$, where m_i stands for the polar point to the middle slice M_i. Assuming that $\langle m_i, m_i\rangle = 1$, we obtain $R_i : x \mapsto 2\langle x, m_i\rangle m_i - x$ by (2.1.3). Let $b = b_1 \cup \cdots \cup b_n$ be a meridional curve of $(\mathrm{B}_1, \ldots, \mathrm{B}_n)$ with vertices $s_1, \ldots, s_n$. Fixing a representative $s_1 \in V$, define $s_{i+1} = R_i s_i \in V$. In particular, $s_{n+1} = \delta s_1$. Since $\langle R_i x, x\rangle = 2\langle x, m_i\rangle\langle m_i, x\rangle - \langle x, x\rangle > 0$ for every $x \in \mathbb{B}$, we obtain $\langle s_{i+1}, s_i\rangle > 0$.

Let $D \subset \mathbb{B}$ be any disc with $\partial D = b$. In a standard way, we define a ϱ-equivariant continuous map $\varphi : \mathbb{H}^2_\mathbb{R} \to \mathbb{B}$ such that $\varphi(P) = D$, $\varphi(v_i) = s_i$, and $\varphi(e_i) = b_i$ (see 2.1.9). The Toledo invariant of ϱ is defined as

$$\tau := \frac{4}{2\pi}\int_P \varphi^* \omega$$

(the hermitian metric we introduced differs by the factor of 4 from the one in [Tol] (see Corollary 4.1.18)). Taking $c \in \mathbb{B}$ and applying Lemma 2.1.12, we obtain

$$\tau = \frac{2}{\pi}\int_D \omega = \frac{2}{\pi}\int_{\partial D} P_c = \frac{2}{\pi}\sum_i \int_{b_i} P_c.$$

By Lemmas 2.1.13 and 2.1.12,

$$\int_{b_i} P_c = \int_{b_i}(P_c - P_{s_i}) = \int_{b_i} df_{c,s_i}.$$

This number is the total variation of $\dfrac{1}{2}\mathrm{Arg}\,\dfrac{\langle c, p\rangle\langle p, s_i\rangle}{\langle c, s_i\rangle}$, while p runs over b_i from s_i to s_{i+1}. By Lemma 2.1.14, $\dfrac{\langle c, p\rangle\langle p, s_i\rangle}{\langle c, s_i\rangle}$ cannot be real nonnegative. It follows that

$$\int_{b_i} df_{c,s_i} = \frac{1}{2}\mathrm{Arg}\,\frac{\langle c, s_{i+1}\rangle\langle s_{i+1}, s_i\rangle}{\langle c, s_i\rangle} - \frac{1}{2}\mathrm{Arg}\,\frac{\langle c, s_i\rangle\langle s_i, s_i\rangle}{\langle c, s_i\rangle} = \frac{1}{2}\mathrm{Arg}\,\frac{\langle c, s_{i+1}\rangle}{\langle c, s_i\rangle} - \frac{\pi}{2} \qquad (\mathbf{2.1.16})$$

since $\langle s_{i+1}, s_i\rangle > 0$ and $\langle s_i, s_i\rangle < 0$. Calculating mod 2, we get

$$\tau \equiv \frac{1}{\pi}\sum_i \big(\mathrm{Arg}\langle c, s_{i+1}\rangle - \mathrm{Arg}\langle c, s_i\rangle - \pi\big) \equiv$$

$$\equiv \frac{1}{\pi}\big(-n\pi + \mathrm{Arg}\langle c, s_{n+1}\rangle - \mathrm{Arg}\langle c, s_1\rangle\big) \equiv -n - \frac{\mathrm{Arg}\,\delta}{\pi} \quad \text{mod 2.} \qquad \blacksquare$$

Considering the surface groups G_n or T_n of finite index in H_n (see 2.1.9), we obtain the following

Corollary 2.1.17. *For even* n, *the Toledo invariant of the induced representation of* G_n *in* PU *satisfies* $\tau \equiv -\dfrac{2\,\mathrm{Arg}\,\delta}{\pi}$ mod 4. *For odd* n, *the Toledo invariant of the induced representation of* T_n *in* PU *satisfies* $\tau \equiv 4n - \dfrac{4\,\mathrm{Arg}\,\delta}{\pi}$ mod 8, *where* δ *is introduced in Definition 2.1.7.*

Proof. When passing to a finite cover, the Toledo invariant gets multiplied by the degree of the cover. In our case, this means that the first integral in Proposition 2.1.15 should be taken over two or four copies of a fundamental domain for H_n. ∎

2.2. Discreteness

We intend to prove discreteness of the group H generated by reflections in middle slices of a cycle of bisectors by showing that a polyhedron bounded by the cycle is fundamental.

All isometries in this subsection are considered as belonging to PU.

2.2.1. Given oriented bisectors B and B$'$ with a common slice S, denote by n and n' the normal vectors at $s \in S \cap \mathbb{B}$ to B and to B$'$, respectively. Since both n and n' are tangent to the naturally oriented complex geodesic passing through s and orthogonal to S, it makes sense to measure the oriented angle from n to n'. This angle is said to be the *oriented angle* from B to B$'$ at s.

Let $(B_1, \ldots, B_n)$ be a cycle of bisectors with $B_i = B[S_i, S_{i+1}]$ (the indices are modulo n). Denote by $B_i^- := B[S_{i+1}, S_i]$ the segment B_i taken with the opposite orientation. Pick a point $s_1 \in S_1 \cap \mathbb{B}$ and define $s_{i+1} := R_i s_i$. Clearly, $s_i \in S_i \cap \mathbb{B}$ and $s_{n+1} = s_1$ because $R_n \ldots R_1 = 1$. Denoting by α_i the oriented angle from B_i to B_{i-1}^- at s_i, we call $\alpha := \alpha_1 + \cdots + \alpha_n$ the *total angle* of the cycle at s_1.

Definition 2.2.2. A simple configuration of bisectors $(B_1, \ldots, B_n)$ gives rise to the closed solid torus $\partial_0 P := B_1 \cup \cdots \cup B_n$ *fibred by slices* that divides $\overline{\mathbb{B}}$ into two closed connected parts. The part P situated on the side of the normal vector to each B_i is called the *polyhedron* of the configuration. The torus $T := \partial_\infty \partial_0 P$ divides the boundary of P into two closed parts, $\partial P = \partial_0 P \cup \partial_\infty P$. So, $T = \partial \partial_\infty P = \partial_0 P \cap \partial_\infty P$. The ideal meridional curves are contained in T. In the particular case of a cycle of bisectors, $\partial_0 P$ is also *fibred by meridional curves* (hence, T is fibred by the ideal meridional curves).

In order to apply [AGr3, Theorem 3.5] to the polyhedron $P \cap \mathbb{B}$, where P is the polyhedron of a simple transversal cycle of bisectors, we equip $P \cap \mathbb{B}$ with the face-pairing isometries R_i, thus pairing each face with itself and getting a single geometric cycle of edges. As in [AGr3], we denote by $N(X, \varepsilon)$ the ε-neighbourhood of $X \subset \mathbb{B}$.

Lemma 2.2.3 [Appendix, Lemma 4.3.6]**.** *Let* B *and* B$'$ *be full bisectors transversal along a common slice* S*. Then, for every* $\vartheta > 0$*, there exists* $\varepsilon > 0$ *such that* $B' \cap N(B \cap \mathbb{B}, \varepsilon) \subset N(S \cap \mathbb{B}, \vartheta)$*.*

Theorem 2.2.4. *Let* $(B_1, \ldots, B_n)$ *be a simple transversal cycle of bisectors with total angle* 2π *at some point and let* P *denote the polyhedron of the cycle. Then the subgroup* $H \subset$ PU *generated by the reflections in the middle slices of the* B_i*'s is discrete and* P *is a fundamental region in* $\mathbb{B}$ *for* H*. The group* H *is isomorphic to* H_n*.*

Proof. The only condition in the definition of cornerless polyhedron [AGr3, Section 3] that is not completely obvious for $P \cap \mathbb{B}$ is Strong Simplicity IV. This condition follows from the fact that B_i and B_j are disjoint in $\overline{\mathbb{B}}$ for $j \notin \{i-1, i, i+1\}$, implying the metric separability of $B_i \cap \mathbb{B}$ and $B_j \cap \mathbb{B}$. In order to verify Condition (1) of [AGr3, Theorem 3.5], it suffices to observe that R_i preserves the complex spine of B_i, where it acts as the reflection in a point in the real spine. It is easy to see that the oriented angle from B_i to B_{i-1}^- at a point $s_i \in S_i \cap \mathbb{B}$ coincides with the oriented interior angle at s_i defined in [AGr3, Section 3.3]. Hence, the single geometric cycle of edges of $P \cap \mathbb{B}$ has total interior angle 2π at some point of an edge and Condition (2) of [AGr3, Theorem 3.5] is satisfied. Condition (3) of [AGr3, Theorem 3.5] is valid by Lemma 2.2.3. ∎

2.3. Simplicity and transversality

In order to apply Theorem 2.2.4, we need a criterion providing simplicity of a cycle. We have found no adequate numerical criterion of this sort since it is rather difficult to decide whether two segments of bisectors intersect or not (see, for instance, [San]). Fortunately, appealing to simple kind of Poincaré duality, simplicity of a cycle can be inferred from some extra transversalities (see Criterion 2.3.9).

All isometries in this subsection are considered as belonging to PU.

2.3.1. Notation concerning bisectors. Every two distinct points $p_1, p_2 \in \overline{\mathbb{B}}$ determine the closed oriented segment of geodesic $G[p_1, p_2]$ from p_1 to p_2 and the corresponding closed oriented segment of bisector $B[p_1, p_2]$ whose real spine is $G[p_1, p_2]$. (Note that we also use to specify a closed oriented segment of bisector in terms of two ultraparallel complex geodesics as in 2.1.1.) For an oriented segment of bisector $B := B[p_1, p_2]$, we denote by $\prec B \succ$ the full bisector of the segment, endowed with the corresponding orientation. Hence, $\prec B \succ = B[u, v]$, where u and v are the vertices of $\prec B \succ$ taken in a suitable order. We also denote $B \succ := B[p_1, v]$ and $B^- := B[p_2, p_1]$ (so, B^- is B taken with the opposite orientation). In this way, we deal with three kinds of segments of bisectors: finite ones, semifinite ones, and infinite ones (full bisectors).
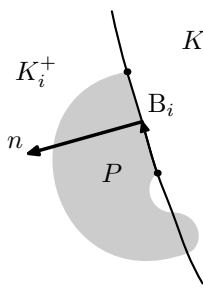
Every full oriented bisector B divides $\overline{\mathbb{B}}$ into two *half-spaces* (closed 4-balls) K^+ and K^-, where K^+ lies on the side of the normal vector to B. Note that, in general, the fundamental polyhedron P in Theorem 2.2.4 is not included in the half-space K_i^+ *bounded* by $\prec B_i \succ$.

Lemma 2.3.2 [Appendix, Lemma 4.3.5]**.** *Let* B *and* B′ *be two full bisectors transversal along a common slice* S*. Then* B*,* B′*, and* $\partial_\infty \mathbb{B}$ *are transversal and* $B \cap B' = S$*.*
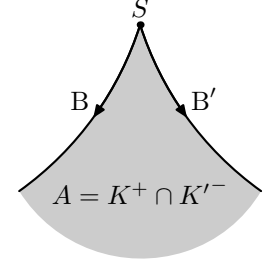
Let B and B′ be two segments of bisectors that begin with a common slice S. Suppose that $\prec B \succ$ and $\prec B' \succ$ are transversal along S. Then, by Lemma 2.3.2, $\prec B \succ$ and $\prec B' \succ$ divide $\overline{\mathbb{B}}$ into four closed 4-balls. We define the *sector* A from $B \succ$ to $B' \succ$ as follows. If the oriented angle from $B \succ$ to $B' \succ$ at a point in S does not exceed π (it does not matter at which point in S we measure the angle), then we put $A := K^+ \cap K'^-$, where K^+ and K'^+ stand respectively for the half-spaces bounded by $\prec B \succ$ and $\prec B' \succ$. Otherwise, we put $A := K^+ \cup K'^-$.
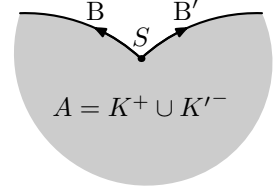
2.3.3. Transversal adjacency. Let C, M_1, M_2 be pairwise ultraparallel complex geodesics. We connect them with the oriented segments of bisectors $B_1 := B[C, M_1]$, $B := B[M_1, M_2]$, and $B_2 := B[M_2, C]$ thus obtaining the *oriented triangle* $\Delta(C, M_1, M_2)$ of bisectors. The triangle is said to be *transversal* if the corresponding full bisectors are transversal along their common slices. Since $\prec B_1 \succ$ and $\prec B_2 \succ$ are transversal along C, either the oriented angle from $B_1 \succ$ to $B_2^- \succ$ is less than π or the oriented angle from $B_2^- \succ$ to $B_1 \succ$ is less than π. We denote by D the sector containing the smaller angle and call it the *interior sector* at C. So, either $D = K_1^+ \cap K_2^+$ or $D = K_1^- \cap K_2^-$, where K_i^+ and K^+ denote the half-spaces bounded by $\prec B_i \succ$ and $\prec B \succ$. Similarly, we define the interior sector D_i between B and B_i at M_i. By altering the orientation of the triangle, we can always assume that $D = K_1^+ \cap K_2^+$.

Given one more complex geodesic S ultraparallel to M_1 and to M_2 such that the triangle $\Delta(S, M_2, M_1)$ is also transversal, denote $B_1' := B[M_1, S]$ and $B_2' := B[S, M_2]$. We say that the triangle $\Delta(S, M_2, M_1)$ is *transversally adjacent* to the triangle $\Delta(C, M_1, M_2)$ if some point in S belongs to D and if the bisectors $\prec B_i \succ$ and $\prec B_i' \succ$ are transversal along M_i for $i = 1, 2$.

Lemma 2.3.4. *Let* $\Delta(C, M_1, M_2)$ *be a transversal triangle of bisector oriented so that* $D = K_1^+ \cap K_2^+$*, where* D *stands for the interior sector at* C*. Then the segment* $B := B[M_1, M_2]$ *is contained in* D*. Moreover, for the interior sector* D_i *at* M_i*, we have* $D_i = K^+ \cap K_i^+$*.*

Suppose that some transversal triangle $\Delta(S, M_2, M_1)$ *is transversally adjacent to* $\Delta(C, M_1, M_2)$*. Then*[4] $\Delta(S, M_2, M_1) \subset D$*.*

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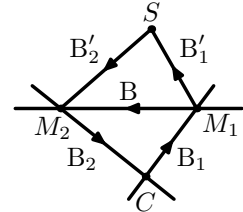
[4]Note that $\Delta(S, M_2, M_1)$ can be 'inside' or 'outside' $\Delta(C, M_1, M_2)$ (see also Remark 2.3.6).

Proof. Since B is connected and intersects $\prec B_i \succ$ only in M_i, the segment B is contained in one of the four sectors formed by the $\prec B_i \succ$'s. The only sector that contains both M_i's is D. This proves that $B \subset D$.

Suppose that $D_1 = K^- \cap K_1^-$. Then, by the above statement, we have $B_2 \subset D_1$, which implies $B_2 \subset K_1^-$. On the other hand, $B_2 \subset D \subset K_1^+$. A contradiction. The same works for D_2. So, $D_i = K^+ \cap K_i^+$.

The bisector $\prec B_1' \succ$ intersects $\prec B_1 \succ$ only in M_1. Hence, either $B_1' \succ \subset K_1^+$ or $B_1' \succ \subset K_1^-$. Since some point of S belongs to $D \subset K_1^+$ and $S \subset B_1' \succ$, we conclude that $B_1' \succ \subset K_1^+$. By the same reason, $B_2'^- \succ \subset K_2^+$. From $B \subset D$, we derive that $B \succ \subset K_1^+$ and $B^- \succ \subset K_2^+$. Consequently, $D_1' \subset K_1^+$ and $D_2' \subset K_2^+$, where D_i' stands for the interior sector of $\Delta(S, M_2, M_1)$ at M_i. Since $B_2' \subset D_1'$ and $B_1' \subset D_2'$, we obtain $B_2' \subset K_1^+$ and $B_1' \subset K_2^+$. Now, from $B_1' \succ \subset K_1^+$ and $B_2'^- \succ \subset K_2^+$, we deduce that $B_i' \subset D$. Therefore, $\Delta(S, M_2, M_1) \subset D$. ∎

Definition 2.3.5. The orientation of the transversal triangle of bisectors $\Delta(C, M_1, M_2)$ dealt with in Lemma 2.3.4 is said to be *counterclockwise*.

Remark 2.3.6. *Let $\Delta(C, M_1, M_2)$ and $\Delta(S, M_2, M_1)$ be counterclockwise-oriented transversal triangles. Suppose that $\Delta(S, M_2, M_1)$ is transversally adjacent to $\Delta(C, M_1, M_2)$. Then $\Delta(S, M_2, M_1)$ and $\Delta(C, M_1, M_2)$ lie in the distinct half-spaces bounded by $\prec B[M_1, M_2] \succ$.* ∎

2.3.7. Centre of a configuration. Let $(B_1, \ldots, B_n)$ be a configuration of bisectors and let C be a complex geodesic ultraparallel to every middle slice M_i. Denote $B_i' := B[C, M_i]$. If, for every i, the full bisectors $\prec B_{i-1}' \succ$ and $\prec B_i' \succ$ are transversal along their common slice C, then we call C a *centre* of the configuration. Take $c \in C$. Denoting by β_i the oriented angle from $B_{i-1}' \succ$ to $B_i' \succ$ at c, it is easy to see that $\beta := \beta_1 + \cdots + \beta_n$, the *central angle* of the configuration at C, does not depend on the choice of $c \in C$ and is an integer multiple of 2π. The condition that a configuration possesses a centre does not seem too restrictive.
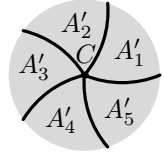
Lemma 2.3.8. *Let B_i', $1 \leq i \leq n$, be segments of bisectors. Suppose that they all have a common slice C (they all begin with C), that $\prec B_{i-1}' \succ$ and $\prec B_i' \succ$ are transversal along C for all i, and that, for some $c \in C \cap \mathbb{B}$, the sum of the angles from $B_{i-1}' \succ$ to $B_i' \succ$ at c equals 2π. Then the $B_i' \succ$'s divide $\overline{\mathbb{B}}$ into n sectors A_i' (from $B_{i-1}' \succ$ to $B_i' \succ$) such that $A_i' \cap A_j' = C$ if $j \notin \{i-1, i, i+1\}$ and $A_i' \cap A_{i+1}' = B_i' \succ$. In other words, there are no extra intersections besides the obvious ones.*

Proof. Since $C \cap \mathbb{B}$ is connected, the mentioned sum of angles equals 2π at every point in $C \cap \mathbb{B}$. Let $p \in A_i' \cap A_j'$ be a point that does not belong to the intersections of A_i' and A_j' listed above. Take the complex geodesic S passing though p and orthogonal to C. By Lemma 2.3.2, inside S, every pairwise intersection of the $B_i' \succ$'s is the point $S \cap C$. Hence, inside S, the sectors A_i''s can only intersect in the obvious way. A contradiction. ∎

If a configuration is transversal, then M_{i-1} and M_i are ultraparallel by Lemma 2.3.2. In this case, assuming that the configuration possesses a centre C, we obtain the oriented triangles $\Delta_i := \Delta(C, M_{i-1}, M_i)$ and $\Delta_i' := \Delta(S_i, M_i, M_{i-1})$.

To a certain extent, transversality implies simplicity:

Criterion 2.3.9. *Let $(B_1, \ldots, B_n)$ be a transversal configuration of bisectors possessing a centre C. Suppose that the triangles Δ_i and Δ_i' are transversal and that Δ_i' is transversally adjacent to Δ_i for every i. If the central angle at some point is 2π and the angle from $B_{i-1}' \succ$ to $B_i' \succ$ does not[5] exceed π for every i, then the cycle is simple.*

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[5]This condition is not too restrictive, since only one of these angles can exceed π.

Proof. By Lemma 2.3.8, the segments $B_i' \succ$ divide $\overline{\mathbb{B}}$ into n sectors A_i''s whose interiors are disjoint. The triangle Δ_i' is transversally adjacent to the triangle Δ_i, therefore, $B_{i-1} \cap B_{i-1}' \succ = M_{i-1}$ and $B_i \cap B_i' \succ = M_i$. By Lemma 2.3.4, $B[M_{i-1}, S_i] \cup B[S_i, M_i] \subset A_i'$. ∎

2.4. Fibred polyhedra. Euler number

Every complex hyperbolic disc bundle in the explicit series of examples we construct in Section 3 is glued from fibred polyhedra. The Euler number of the resulting bundles will be derived from 'fractional Euler quantities' of these parts.

We work in the PL category.

2.4.1. Fibred polyhedra. Let P be the polyhedron of a simple configuration of bisectors (see Definition 2.2.2). In the situation described in the next lemma, P is said to be *fibred*.

Lemma 2.4.2. *Suppose that the polyhedron P of a simple configuration of bisectors is a closed 4-ball, $P \simeq \mathbf{B}^4$. Then $P \simeq \mathbf{B}^2 \times \mathbf{B}^2$ is a bundle with $\partial_0 P \simeq \mathbb{S}^1 \times \mathbf{B}^2$ being a subbundle if and only if $\partial_\infty P$ is a solid torus. In this case, the slice bundle structure of $\partial_0 P$ is extendable to P.*

Proof. The arguments are standard. If $P \simeq \mathbf{B}^2 \times \mathbf{B}^2$, then $P \simeq \mathbf{B}^4$ and ∂P is a sphere $\mathbb{S}^3$ decomposed into two solid tori $\partial \mathbf{B}^2 \times \mathbf{B}^2$ and $\mathbf{B}^2 \times \partial \mathbf{B}^2$ glued along the torus $\partial \mathbf{B}^2 \times \partial \mathbf{B}^2$. Hence, $\partial_0 P \simeq \partial \mathbf{B}^2 \times \mathbf{B}^2$ implies that $\partial_\infty P \simeq \mathbf{B}^2 \times \partial \mathbf{B}^2$.

Conversely, if $\partial_\infty P$ is a solid torus, then $\partial P \simeq \mathbb{S}^3$ is decomposed into two solid tori glued along the torus T. As is well known, such a decomposition of $\mathbb{S}^3$ is topologically unique. Arbitrarily fibering one of the solid tori by discs, we fiber T by circles. Extending the latter circle bundle, we fiber the other solid torus by circles and obtain the compatible decompositions $T \simeq \mathbb{S}^1 \times \mathbb{S}^1$, $\partial_0 P \simeq \mathbb{S}^1 \times \mathbf{B}^2$, and $\partial_\infty P \simeq \mathbf{B}^2 \times \mathbb{S}^1$. Since $P \simeq \mathbf{B}^4$ is a cone over $\partial P \simeq \mathbb{S}^3$, we can readily extend these decompositions to a compatible decomposition $P \simeq \mathbf{B}^2 \times \mathbf{B}^2$. ∎

The Dehn Lemma immediately implies the

Remark 2.4.3. *Let P be the polyhedron of a simple configuration of bisectors. Then P is fibred (hence, $P \simeq \mathbf{B}^4$) if and only if there exists some simple closed curve $c \subset T$ contractible in $\partial_\infty P$ such that c intersects each slice of $\partial_0 P$ exactly once, i.e., $[c]$ generates $\pi_1(\partial_0 P)$.* ∎

The curve c in Remark 2.4.3 is called *trivializing*.

2.4.4. Gluing fibred polyhedra. Let $(B_1, \ldots, B_i, B_{i+1}, \ldots, B_n)$, $1 \le i < n$, and $(B_m', \ldots, B_{i+1}', B_i^-, \ldots, B_1^-)$, $1 \le i < m$, be simple configurations of bisectors with a common sequence of bisectors (the segments common to both configurations have opposite orientations) such that the polyhedra P_1 and P_2 of the configurations intersect only in $B_1 \cup \cdots \cup B_i$. Then we can glue P_1 and P_2 along $B_1 \cup \cdots \cup B_i$, obtaining the *gluing* $P := P_1 \cup P_2$ which is the polyhedron of the simple configuration $(B_{i+1}, \ldots, B_n, B_m', \ldots, B_{i+1}')$. The polyhedra P and $R_j P$ in Theorem 2.2.4 yield an example of such a gluing: they are glued along B_j. Suppose that P_1 and P_2 are fibred. Then the solid tori $\partial_\infty P_1$ and $\partial_\infty P_2$ intersect in the annulus $\partial_\infty B_1 \cup \cdots \cup \partial_\infty B_i$ which is an annular neighbourhood of a simple curve generating the fundamental group of each
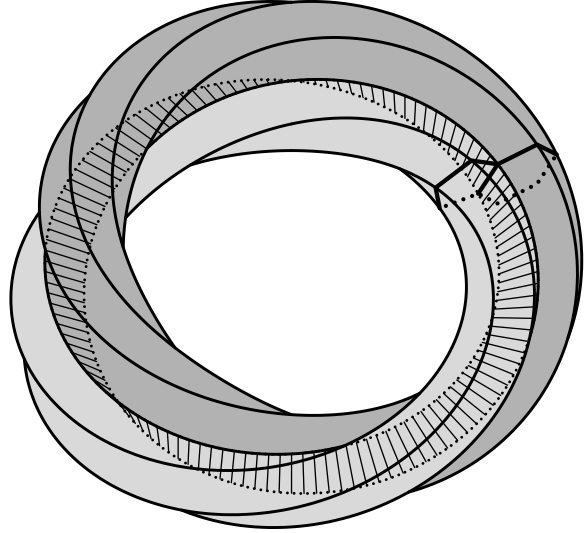
solid torus $\partial_\infty P_k$, $k = 1, 2$. Hence, we can choose a trivializing curve c_k contractible in $\partial_\infty P_k$ so that c_1 and c_2 coincide along the annulus. Thus, we arrive at the

Remark 2.4.5. *Let P_1 and P_2 be fibred polyhedra and let $P := P_1 \cup P_2$ be their gluing. Then P is fibred and a trivializing curve of P can be obtained by gluing (and removing the common part from) trivializing curves c_1 of P_1 and c_2 of P_2 which coincide along the common sequence of bisectors.* ∎

2.4.6. Euler number. Let P be a fibred polyhedron of a simple cycle, equipped with the face-pairing given by reflections in middle slices, and let b be a meridional curve of the cycle. Clearly, there exists a simple closed disc $D \subset P$ that intersects each fibre in P exactly once and such that $b = \partial D$ and $\mathring{D} \subset \mathring{P}$ (where $\mathring{X}$ denotes the interior of X). We assume that meridional curves and trivializing curves are *oriented* with respect to the orientation of the cycle. We *orient* the twice fibred torus T as follows: the first coordinate is the naturally oriented boundary of a slice and the second is an ideal meridional curve, already oriented. The orientation of b *orients* D. We call D an *equivariant section* of the fibred polyhedron P with respect to reflections in the middle slices. For another meridional curve b' (clearly, $b \cap b' = \varnothing$), we can find an equivariant section D' with $\partial D' = b'$ and choose D and D' to be transversal. We call the number $e_P := \#D \cap D'$ (taking signs into account) the *Euler number*[6] of the polyhedron P equipped with its face-pairing (see also 2.4.8).

Clearly, e_P measures the difference between two decompositions of the slice bundle $\partial_0 P$ into the product $\mathbb{S}^1 \times \mathbf{B}^2$: the decomposition into meridional curves and the decomposition induced by the trivialization $P \simeq \mathbf{B}^2 \times \mathbf{B}^2$ (in the torus T, fibred by the boundary of slices, the trivialization $P \simeq \mathbf{B}^2 \times \mathbf{B}^2$ induces a bundle of trivializing curves). Obviously, such a difference can be measured in terms of T :

Remark 2.4.7. *Let P be a fibred polyhedron of a simple cycle. Then $e_P = \#b \cap c$, where b stands for an ideal meridional curve and c, for a trivializing curve. In other words, $[b] = e_P \cdot [s]$ in the group $H_1(\partial_\infty P, \mathbb{Z})$ generated by $[s]$, where $[s]$ is represented by the boundary of a naturally oriented slice.* ∎

2.4.8. Let P be a fibred polyhedron of a cycle of bisectors that satisfies the conditions of Theorem 2.2.4. Then the Euler number of the complex hyperbolic disc bundle associated to the corresponding discrete representation of G_n or T_n equals respectively $2e_P$ or $4e_P$ (see 2.1.9) since the Euler number of a polyhedron is additive with respect to gluing and the face-pairing respects the meridional curves. Actually, e_P is the Euler number of the orbifold disc bundle over $\mathbb{S}^2(2, \dots, 2)$ (see [Fen] for a definition). The Euler number of an orbifold is known to be additive under finite covers.

Proposition 2.4.9. *Let $(B_1, \dots, B_n)$ be a simple transversal cycle of bisectors with total angle 2π at some point and suppose that the polyhedron P of the cycle is fibred. Then, for odd n, the complex hyperbolic manifold $\mathbb{B}/T_n$ is diffeomorphic to a disc bundle over a closed orientable surface of genus $n - 3$ with Euler number $4e_P$. For even n, the complex hyperbolic manifold $\mathbb{B}/G_n$ is diffeomorphic to a disc bundle over a closed orientable surface of genus $\frac{n}{2} - 1$ with Euler number $2e_P$.* ∎

2.5. Transversal triangles

Transversal triangles of bisectors turn out to be important geometrical objects. They can serve as building blocks for constructing (fibred) fundamental polyhedra, and not only in the way used in this paper. They are naturally equipped with an isometry of their vertices (complex geodesics), called holonomy. In this subsection, we prove that every transversal triangle provides a fibred polyhedron (Theorem 2.5.2).

2.5.1. Holonomy of a triangle. Let B be an oriented full bisector. The meridional curves induce an identification of the slices in B called the *slice identification along* B. By Lemma 2.1.4, for an oriented

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[6]Standard arguments show that this number does not depend on the choice of the meridional curves b and b' and of the equivariant sections D and D'.

segment $B[S, S']$, the slice identification between S and S' is induced by the reflection in the middle slice of $B[S, S']$. Therefore, this identification is an isometry between the slices. Clearly, such an identification can be also described via a suitable one-parameter subgroup in PU.

Let (B_1, B_2, B_3) be a counterclockwise-oriented transversal triangle of bisectors (see Definition 2.3.5). Denote by R_i the reflection in the middle slice of B_i and let S_1 stand for the initial slice of B_1. We put $\varphi := R_3 R_2 R_1$. Consider the identification of S_1 with itself given by the slice identification, first, along B_1, then along B_2, and, finally, along B_3. This identification is obviously induced by $\varphi \in PU$, $\varphi S_1 = S_1$. We call φ the *holonomy* of the triangle (B_1, B_2, B_3). The triangle is said to be *elliptic, parabolic, hyperbolic,* or *trivial* if φ, being restricted to S_1, is elliptic, parabolic, hyperbolic, or the identity.

Suppose that the triangle is elliptic. For every $s_1 \in S_1$, following the counterclockwise orientation of the triangle, we can draw a meridional curve b that begins with s_1. This curve ends with some $s_1' \in S_1$. In other terms, $\varphi s_1 = s_1'$. We take $s_1 \in \partial_\infty S_1$. Then $s_1' \in \partial_\infty S_1$. Following the natural orientation of the circle $\partial_\infty S_1$, we can draw an arc $a \subset \partial_\infty S_1$ from s_1' to s_1 obtaining a closed oriented curve $c := b \cup a \subset T$, where T stands for the torus of the triangle. The curve c is said to be *standard*.

In the case of a hyperbolic triangle, there are two points in $\partial_\infty S_1$ fixed by φ. They divide $\partial_\infty S_1$ into two φ-invariant parts: the R-*part* where φ moves the points in the counterclockwise direction and the L-*part* where φ moves the points in the clockwise direction. Let $s_1 \in \partial_\infty S_1$ be a point in the interior of the L-part. As above, we can draw a meridional curve b beginning with s_1 and ending with $s_1' = \varphi s_1$ $\in \partial_\infty S_1$. Obviously, s_1' is also in the L-part. Again, we draw an arc a from s_1' to s_1 in the counterclockwise direction. Clearly, a is contained in the L-part of $\partial_\infty S_1$. We call the closed oriented curve $c := b \cup a \subset T$ *standard* as well. Note that there are two closed meridional curves in T (they correspond to the fixed points of φ), both isotopic to a standard one.
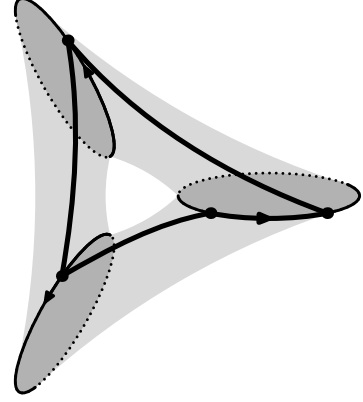
For a parabolic triangle, we distinguish the R-*parabolic* and L-*parabolic* cases. Exactly one point in $\partial_\infty S_1$ is fixed by φ. The isometry φ moves all the other points in $\partial_\infty S_1$ in the same direction, counterclockwise for an R-parabolic triangle and clockwise for an L-parabolic triangle. As above, we define a *standard* curve for an L-parabolic triangle. In T, this curve is isotopic to the closed meridional curve. In the case of an R-parabolic or trivial triangle, we define no standard curve.

We now extend the definition of L-*part* of $\partial_\infty S_1$ to the case of elliptic, parabolic, or trivial triangle. For an elliptic triangle, it is the entire $\partial_\infty S_1$. For an L-parabolic triangle, it is $\partial_\infty S_1$ without the fixed point of φ. For the other two cases, it is empty.

Counterclockwise-oriented transversal triangles of bisectors are simple configurations by Lemma 2.3.2. Hence, they fall into Definition 2.2.2. In order to prove that a polyhedron glued from transversal triangles is fibred and to be able to calculate its Euler number, we need the following

Theorem 2.5.2. *Let* (B_1, B_2, B_3) *be a counterclockwise-oriented transversal triangle of bisectors. Then the triangle cannot be trivial nor* R-*parabolic. The polyhedron* P *of the triangle is fibred and its standard curve is trivializing.*

The proof of Theorem 2.5.2 is postponed until the end of this subsection.

Lemma 2.5.3 [Appendix, Lemma 4.4.18]**.** *Let* C *be a complex geodesic and let* $\Delta_0 := \Delta(p_1, p_2, p_3)$ $\subset C$ *be a counterclockwise-oriented geodesic triangle,* $p_1, p_2, p_3 \in C \cap \mathbb{B}$. *Denote by* S_1 *the complex geodesic passing through* p_1 *and orthogonal to* C *and by* R_i, *the reflection in the complex geodesic passing through the middle point of* $G[p_i, p_{i+1}]$ *and orthogonal to* C *(the indices are modulo 3). Then the isometry* $\varphi := R_3 R_2 R_1$, *being restricted to* S_1, *is a rotation about* p_1 *by the angle* $-2 \operatorname{Area} \Delta_0$.

Lemma 2.5.4. *Theorem* 2.5.2 *holds for every triangle of bisectors with common complex spine.*

Proof. The first assertion in Theorem 2.5.2, in the case of common complex spine, follows from Lemma 2.5.3. It remains to prove the second assertion.

Every triangle of bisectors with common complex spine C is built over a usual geodesic triangle $\Delta_0 := \Delta(p_1, p_2, p_3) \subset C \simeq \mathbb{H}^1_{\mathbb{C}}$. (The corresponding triangle of bisectors is simply the complex projective cone over Δ_0 intersected with $\overline{\mathbb{B}}$; the vertex of the cone is the polar point to C.) Clearly, Δ_0 is counterclockwise-oriented. Therefore, $\text{Area}\,\Delta_0 \in (0, \frac{\pi}{4})$ (the metric we use differs by the factor of 4 from Poincaré's one (see Corollary 4.1.18)). By Lemmas 2.1.4 and 2.5.3, the restriction of φ to the slice S_1 is a rotation about p_1 by the angle $-2\,\text{Area}\,\Delta_0$. Hence, $\ell a := 2\,\text{Area}\,\Delta_0 \in (0, \frac{\pi}{2})$ is the angular measure (with respect to the centre p_1) of the arc a ending with $s_1 \in \partial_\infty S_1$. We can contract Δ_0 to p_1 inside Δ_0. We put $\Delta_t := \Delta\big(p_1, p_2(t), p_3(t)\big)$, $t \in [0,1]$, where $p_i(t) \in \text{G}[p_i, p_1]$, $p_i(0) = p_i$, and $p_i(1) = p_1$, $i = 2, 3$. For every $t \in [0,1]$, the triangle $\mathcal{C}_t$ of bisectors built over Δ_t contains the slice S_1. We draw the standard curve c_t of $\mathcal{C}_t$ that starts with s_1. By Lemma 2.1.5, the meridional curve depends continuously on its initial point and on the bisector involved. Since $\partial_\infty P_t \subset \partial_\infty P_0$ and $\ell a_t \to 0$ while $t \to 1$, we can see now that c_0 contracts to s_1 inside $\partial_\infty P_0$. ∎

2.5.5. Let $\Delta(S_1, S_2, S_3)$ be an oriented triangle of bisectors. Denote by g_i the polar point to S_i, $i = 1, 2, 3$, and define

$$t_{ij} := \sqrt{\text{ta}(g_i, g_j)} \ \text{ for } i \neq j, \quad \varkappa := \frac{\langle g_1, g_2\rangle\langle g_2, g_3\rangle\langle g_3, g_1\rangle}{\langle g_1, g_1\rangle\langle g_2, g_2\rangle\langle g_3, g_3\rangle}, \quad \varepsilon := \frac{\varkappa}{|\varkappa|}, \quad \varepsilon_0 := \text{Re}\,\varepsilon, \quad \varepsilon_1 := \text{Im}\,\varepsilon.$$

The fact that the S_i's are ultraparallel implies $t_{ij} > 1$. It follows from Sylvester's Criterion (see [KoM, p. 113]) that the determinant of the Gram matrix of g_1, g_2, g_3 is nonpositive, that is, $1 + 2t_{12}t_{23}t_{31}\varepsilon_0 \leq t_{12}^2 + t_{23}^2 + t_{31}^2$, and that the equality occurs exactly when g_1, g_2, g_3 are in the same projective line. The same criterion immediately implies that the numbers t_{ij} and ε constitute a complete set of geometric invariants of an oriented triangle and that there exists an oriented triangle corresponding to any given values of t_{ij}'s and ε subject to the conditions $t_{ij} > 1$, $|\varepsilon| = 1$, and $1 + 2t_{12}t_{23}t_{31}\varepsilon_0 \leq t_{12}^2 + t_{23}^2 + t_{31}^2$.

Criterion 2.5.6 [Appendix, Criterion 4.4.2]. *Suppose that* $1 < t_{12} \leq t_{23}, t_{31}$. *Then the triangle* $\Delta(S_1, S_2, S_3)$ *is transversal if and only if* $t_{12}^2\varepsilon_0^2 + t_{23}^2 + t_{31}^2 < 1 + 2t_{12}t_{23}t_{31}\varepsilon_0$. *A transversal triangle is counterclockwise-oriented if and only if* $\varepsilon_1 < 0$.

Lemma 2.5.7 [Appendix, Lemma 4.4.3]. *The space of all counterclockwise-oriented transversal triangles of bisectors, i.e., the region in* $\mathbb{R}^4$ *given by the inequalities*

$$|\varepsilon_0| < 1, \qquad 1 < t_{12} \leq t_{23}, t_{31}, \qquad t_{12}^2\varepsilon_0^2 + t_{23}^2 + t_{31}^2 < 1 + 2t_{12}t_{23}t_{31}\varepsilon_0 \leq t_{12}^2 + t_{23}^2 + t_{31}^2$$

is path-connected.

Lemma 2.5.8 [Appendix, Lemma 4.4.15]. *A counterclockwise-oriented transversal triangle of bisectors cannot be* R-*parabolic nor trivial.*

Proof of Theorem 2.5.2. The first assertion in the theorem corresponds to Lemma 2.5.8. We will prove the second assertion using a deformation of the parameters $\varepsilon_0, t_{12}, t_{23}, t_{31}$ of a given counterclockwise-oriented transversal triangle of bisectors. By Criterion 2.5.6, $\varepsilon := \varepsilon_0 - i\sqrt{1 - \varepsilon_0^2}$. By Lemma 2.5.7, we can always reach a triangle of the type dealt with in Lemma 2.5.4. This continuous deformation of a triangle induces a continuous deformation of the holonomy φ, of the nonordered pair of the fixed points of φ in the projective line of S_1, and, hence, of the L-part of $\partial_\infty S_1$ (nonempty by Lemma 2.5.8). Now we can choose s_1 in the L-part of $\partial_\infty S_1$ so that s_1 varies continuously during the deformation. The point s_1 generates a standard curve c. Since, by Lemma 2.1.5, the meridional curve depends continuously on its initial point and on the bisector involved, the curve c varies continuously during the deformation.

In other words, we obtain an isotopy of a 3-manifold F ($= \partial_\infty P$) in $\mathbb{S}^3$ ($= \partial_\infty \mathbb{B}$) equipped with a simple curve ($= c$) on its boundary ($= \partial\partial_\infty P$) which is a simple torus ($= T$). At the end of this isotopy, F is a solid torus and the curve on its boundary is contractible in F. Therefore, the initial polyhedron $\partial_\infty P$ is a solid torus and its standard curve is contractible. ∎

3. Series of explicit examples

In this section, we construct an explicit series of discrete groups with generators U, W and defining relations $U^n = W^n = (U^{-1}W)^2 = 1$. Taking a subgroup of finite index, we arrive at a disc bundle over a closed orientable surface. Then we calculate the Toledo invariant and find the Euler number of these bundles. Finally, we list and discuss many particular examples.

3.1. A couple of transversal triangles

Following the line of Section 2, we construct two counterclockwise-oriented transversal triangles $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$ such that $\Delta(S_2, M_2, M_1)$ is transversally adjacent to $\Delta(C, M_1, M_2)$. Such triangles will be used as building blocks for fundamental polyhedra. During this subsection, we elaborate some explicit and easily verifiable conditions, called quadrangle conditions (see 3.1.17), providing the desired properties of the triangles in question (such as those listed in Proposition 3.1.18).

All isometries in this subsection are considered as belonging to SU.

3.1.1. First, we look for a regular elliptic isometry $W \in$ SU (see [Gol, p. 203] for the definition) and for a reflection $R := R(m) \in$ SU in a complex geodesic $M_1 := \mathbb{P}m^\perp \cap \overline{\mathbb{B}}$ such that $U := WR$ is a regular elliptic isometry. In what follows, the orthonormal basis $q_1, q_2, q_3 \in V$ of signature $- + +$ is formed by the eigenvectors of W. In this basis,

$$m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad R = \begin{pmatrix} -2m_1^2-1 & 2m_1m_2 & 2m_1m_3 \\ -2m_1m_2 & 2m_2^2-1 & 2m_2m_3 \\ -2m_1m_3 & 2m_2m_3 & 2m_3^2-1 \end{pmatrix}, \qquad W = \begin{pmatrix} w_1^2 & 0 & 0 \\ 0 & w_2^2 & 0 \\ 0 & 0 & w_3^2 \end{pmatrix},$$

where m_1, m_2, m_3 will be nonnegative real numbers such that $\langle m, m \rangle = 1$ and the matrix form of R is easily derivable from (2.1.3). We will denote by h_1, h_2, h_3 the eigenvectors of U and by u_1^2, u_2^2, u_3^2 the corresponding eigenvalues.

Take $k, l, n, p \in \mathbb{Z}$ subject to the conditions

$$0 \le k \le l \le n - 3, \qquad p = 1, 2 \tag{3.1.2}$$

and define

$$u_1 := \exp\frac{(2np - k)\pi i}{3n}, \qquad u_2 := \exp\frac{(2np - k - 3)\pi i}{3n}, \qquad u_3 := \exp\frac{(2np + 2k + 3)\pi i}{3n},$$

$$w_1 := \exp\frac{l\pi i}{3n}, \quad w_2 := \exp\frac{(l+3)\pi i}{3n}, \quad w_3 := \exp\frac{-(2l+3)\pi i}{3n}, \quad v := \frac{u_1^2 + u_2^2 + u_3^2 + w_1^2 + w_2^2 + w_3^2}{2}.$$

Observe the following straightforward facts: u_1^2, u_2^2, u_3^2 are pairwise distinct; w_1^2, w_2^2, w_3^2 are pairwise distinct; $u_1 u_2 u_3 = w_1 w_2 w_3 = 1$; $u_1^{2n} = u_2^{2n} = u_3^{2n}$; and $w_1^{2n} = w_2^{2n} = w_3^{2n}$.

The inequalities $0 \le l \le n-3$ imply the inequalities $0 < \operatorname{Re}\big((w_1^2 - w_2^2)w_2\big)$ and $0 < \operatorname{Re}\big((w_1^2 - w_3^2)w_3\big)$. Requiring that

$$\operatorname{Re}(vw_2) < \operatorname{Re} w_2^3, \qquad \operatorname{Re}(vw_3) \le \operatorname{Re}(w_1^2 w_3), \tag{3.1.3}$$

we obtain $\operatorname{Re}\big((w_1^2 - w_2^2)w_2\big) < \operatorname{Re}\big((w_1^2 - v)w_2\big)$ and $0 \le \operatorname{Re}\big((w_1^2 - v)w_3\big)$. Therefore, we can define

$$m_2 := \sqrt{\frac{\operatorname{Re}\big((w_1^2 - v)w_2\big)}{\operatorname{Re}\big((w_1^2 - w_2^2)w_2\big)}}, \qquad m_3 := \sqrt{\frac{\operatorname{Re}\big((w_1^2 - v)w_3\big)}{\operatorname{Re}\big((w_1^2 - w_3^2)w_3\big)}}, \qquad m_1 := \sqrt{m_2^2 + m_3^2 - 1}$$

such that $m_1 > 0$ and $m_2 > 1$.

Lemma 3.1.4. *Under the conditions (3.1.2) and (3.1.3), we have* $\operatorname{tr}(WR) = u_1^2 + u_2^2 + u_3^2$.

Proof. The equality in the lemma has the form

$$(-2m_2^2 - 2m_3^2 + 1)w_1^2 + (2m_2^2 - 1)w_2^2 + (2m_3^2 - 1)w_3^2 = u_1^2 + u_2^2 + u_3^2$$

which is equivalent to

$$m_2^2(w_2^2 - w_1^2) + m_3^2(w_3^2 - w_1^2) = v - w_1^2.$$

By definition,

$$m_2^2 = \frac{w_2(v - w_1^2) + \overline{w}_2(\overline{v} - \overline{w}_1^2)}{w_2(w_2^2 - w_1^2) + \overline{w}_2(\overline{w}_2^2 - \overline{w}_1^2)} =$$

$$= \frac{\left(w_2^2(v - w_1^2) + (\overline{v} - \overline{w}_1^2)\right)w_1^2 w_2^2}{w_1^2 w_2^4(w_2^2 - w_1^2) + (w_1^2 - w_2^2)} = \frac{w_1^2 w_2^4(v - w_1^2) + w_1^2 w_2^2(\overline{v} - \overline{w}_1^2)}{(w_1^2 w_2^4 - 1)(w_2^2 - w_1^2)}.$$

Similarly,

$$m_3^2 = \frac{w_1^2 w_3^4(v - w_1^2) + w_1^2 w_3^2(\overline{v} - \overline{w}_1^2)}{(w_1^2 w_3^4 - 1)(w_3^2 - w_1^2)}.$$

It follows from $w_1 w_2 w_3 = 1$ that

$$\frac{w_1^2 w_2^2}{w_1^2 w_2^4 - 1} + \frac{w_1^2 w_3^2}{w_1^2 w_3^4 - 1} = 0, \qquad \frac{w_1^2 w_2^4}{w_1^2 w_2^4 - 1} + \frac{w_1^2 w_3^4}{w_1^2 w_3^4 - 1} = 1.$$

Now we can see that $(w_2^2 - w_1^2)m_2^2 + (w_3^2 - w_1^2)m_3^2 = v - w_1^2$. ∎

Obviously $R, W \in \mathrm{SU}$. Hence, $U := WR \in \mathrm{SU}$. By [Gol, p. 204, Theorem 6.2.4] and Lemma 3.1.4, U is a regular elliptic isometry with eigenvalues u_1^2, u_2^2, u_3^2. It follows from $u_1^{2n} = u_2^{2n} = u_3^{2n}$ and $w_1^{2n} = w_2^{2n} = w_3^{2n}$ that U^n and W^n belong to the centre of SU.

3.1.5. Let us find the eigenvector h_i of U corresponding to u_i^2. In order to exclude trivial cases, we require that

$$u_i^2 + w_j^2 \neq 0 \quad \text{for all } i, j. \tag{3.1.6}$$

Lemma 3.1.7. *Under the conditions (3.1.2), (3.1.3), and (3.1.6),* $\begin{pmatrix} \frac{m_1}{u_i^2 w_1^{-2} + 1} \\ \frac{m_2}{u_i^2 w_2^{-2} + 1} \\ \frac{m_3}{u_i^2 w_3^{-2} + 1} \end{pmatrix}$ *is the eigenvector of* U *corresponding to* u_i^2.

Proof. Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq 0$. We fix some $i = 1, 2, 3$. The condition that x is the eigenvector of $U = WR$ corresponding to u_i^2 is given by the equalities $2w_j^2 m_j f(x) = (u_i^2 + w_j^2)x_j$, $j = 1, 2, 3$, where $f(x) := -m_1 x_1 + m_2 x_2 + m_3 x_3$. Thus, the inequalities (3.1.6) imply $f(x) \neq 0$ and we can take $x_j = \dfrac{m_j}{u_i^2 w_j^{-2} + 1}$, $j = 1, 2, 3$. ∎

3.1.8. In order to introduce the desired transversally adjacent triangles, we put

$$h_1 := \begin{pmatrix} \frac{m_1}{u_1^2 w_1^{-2} + 1} \\ \frac{m_2}{u_1^2 w_2^{-2} + 1} \\ \frac{m_3}{u_1^2 w_3^{-2} + 1} \end{pmatrix}, \quad h_2 := \begin{pmatrix} \frac{m_1}{u_2^2 w_1^{-2} + 1} \\ \frac{m_2}{u_2^2 w_2^{-2} + 1} \\ \frac{m_3}{u_2^2 w_3^{-2} + 1} \end{pmatrix}, \quad M_2 := WM_1, \quad C := \mathbb{P}q_2^\perp \cap \overline{\mathbb{B}}, \quad S_2 := \mathbb{P}h_2^\perp \cap \overline{\mathbb{B}}$$

and require

$$\langle h_1, h_1 \rangle < 0, \qquad 1 < \mathrm{ta}(m, Wm), \qquad 1 < \mathrm{ta}(m, h_2). \tag{3.1.9}$$

As is well known, the eigenvectors of a regular elliptic isometry are pairwise orthogonal. Hence, only one of the h_i's can be negative, implying that $h_2 \notin \overline{\mathbb{B}}$. So, S_2 is a complex geodesic and $h_1 \in S_2$. Obviously, Wm is the polar point to M_2. The last two inequalities in (3.1.9) respectively mean that the complex geodesics M_1 and M_2 are ultraparallel and that the complex geodesics M_1 and S_2 are ultraparallel [Gol, p. 100]. Since $\langle q_2, m \rangle = m_2$, we obtain $\mathrm{ta}(q_2, m) = m_2^2 > 1$ and conclude that the complex geodesics C and M_1 are ultraparallel as well. Now, it follows from $WC = C$ and from $M_2 = WM_1$ that C and M_2 are ultraparallel. Also, from $RM_1 = M_1$ and $U = WR$, we deduce that $M_2 = WM_1 = WRM_1 = UM_1$. The facts that M_1 and S_2 are ultraparallel, that $M_2 = UM_1$, and that $S_2 = US_2$ imply that M_2 and S_2 are ultraparallel. Thus, we get the triangles $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$.

The points q_2, m, h_2, Wm are polar to the complex geodesics C, M_1, S_2, M_2. The equalities $M_2 = UM_1 = WM_1$, $US_2 = S_2$, and $WC = C$ imply that $\mathrm{ta}(q_2, m) = \mathrm{ta}(Wm, q_2)$ and $\mathrm{ta}(h_2, Wm) = \mathrm{ta}(m, h_2)$.

We introduce the invariants

$$t_{12} := t_{31} := \sqrt{\mathrm{ta}(q_2, m)} = \sqrt{\mathrm{ta}(Wm, q_2)}, \quad t'_{12} := t'_{31} := \sqrt{\mathrm{ta}(h_2, Wm)} = \sqrt{\mathrm{ta}(m, h_2)},$$

$$t_{23} := t'_{23} := \sqrt{\mathrm{ta}(m, Wm)},$$

$$\varkappa := \langle q_2, m \rangle \langle m, Wm \rangle \langle Wm, q_2 \rangle, \qquad \varepsilon := \frac{\varkappa}{|\varkappa|}, \qquad \varepsilon_0 := \mathrm{Re}\,\varepsilon, \qquad \varepsilon_1 := \mathrm{Im}\,\varepsilon,$$

$$\varkappa' := \langle h_2, Wm \rangle \langle Wm, m \rangle \langle m, h_2 \rangle, \qquad \varepsilon' := \frac{\varkappa'}{|\varkappa'|}, \qquad \varepsilon'_0 := \mathrm{Re}\,\varepsilon', \qquad \varepsilon'_1 := \mathrm{Im}\,\varepsilon'$$

of the oriented triangles $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$ (see 2.5.5) and require that

$$t_{12}^2 \varepsilon_0^2 + t_{23}^2 + t_{31}^2 < 1 + 2 t_{12} t_{23} t_{31} \varepsilon_0, \qquad t_{23}^2 \varepsilon_0^2 + t_{31}^2 + t_{12}^2 < 1 + 2 t_{23} t_{31} t_{12} \varepsilon_0, \qquad \varepsilon_1 < 0, \tag{3.1.10}$$

$$t_{12}'^2 \varepsilon_0'^2 + t_{23}'^2 + t_{31}'^2 < 1 + 2 t_{12}' t_{23}' t_{31}' \varepsilon_0', \qquad t_{23}'^2 \varepsilon_0'^2 + t_{31}'^2 + t_{12}'^2 < 1 + 2 t_{23}' t_{31}' t_{12}' \varepsilon_0', \qquad \varepsilon_1' < 0. \tag{3.1.11}$$

Since $t_{12} = t_{31}$ and $t'_{12} = t'_{31}$, both triangles $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$ are transversal and counterclockwise-oriented by Criterion 2.5.6.

Criterion 3.1.12 [Appendix, Criterion 4.3.3]. *Let C, C_1, C_2 be complex geodesics and let g, g_1, g_2 be their polar points. Suppose that C is ultraparallel to C_i for all $i = 1, 2$. Then the full bisectors $\prec \mathrm{B}[C, C_1] \succ$ and $\prec \mathrm{B}[C, C_2] \succ$ are transversal along their common slice C if and only if*

$$\left| \mathrm{Re}\, \frac{\langle g_1, g_2 \rangle \langle g, g \rangle}{\langle g_1, g \rangle \langle g, g_2 \rangle} - 1 \right| < \sqrt{1 - \frac{1}{\mathrm{ta}(g, g_1)}} \cdot \sqrt{1 - \frac{1}{\mathrm{ta}(g, g_2)}}.$$

In the following lemma, we assume that Arg takes values in $[0, 2\pi)$.

Lemma 3.1.13 [Appendix, Corollary 4.3.2]. *Let $\mathcal{E} \in \mathrm{SU}$ be a regular elliptic isometry and let $e_1, e_2, e_3 \in \mathbb{P}$ be the points corresponding to the eigenvectors of $\mathcal{E}$ such that $e_1 \in \mathbb{B}$. Denote by E the complex geodesic with the polar point e_2. Let D be a complex geodesic ultraparallel to E. Then the oriented angle from $\mathrm{B}[E, D]$ to $\mathrm{B}[E, \mathcal{E}D]$ at $e_1 \in E$ equals $\mathrm{Arg}(\xi_2 \xi_1^{-1})$, where ξ_i stands for the eigenvalue of $\mathcal{E}$ corresponding to e_i, $i = 1, 2$.*

It follows from $M_2 = WM_1$, $M_1 = U^{-1}M_2$, and Lemma 3.1.13 that both the angle from $\mathrm{B}[C, M_1]$ to $\mathrm{B}[C, M_2]$ at $q_1 \in C$ and the angle from $\mathrm{B}[S_2, M_2]$ to $\mathrm{B}[S_2, M_1]$ at $h_1 \in S_2$ equal $\dfrac{2\pi}{n}$.

We require that

$$\left| \mathrm{Re}\, \frac{\langle h_2, q_2 \rangle \langle m, m \rangle}{\langle h_2, m \rangle \langle m, q_2 \rangle} - 1 \right| < \sqrt{1 - \frac{1}{\mathrm{ta}(m, h_2)}} \cdot \sqrt{1 - \frac{1}{\mathrm{ta}(m, q_2)}} \tag{3.1.14}$$

(although it is possible to imply (3.1.14) from (3.1.2) and (3.1.3)). By Criterion 3.1.12, the full bisectors $\prec \mathrm{B}[M_1, C]\succ$ and $\prec \mathrm{B}[M_1, S_2]\succ$ are transversal along their common slice M_1. It follows from $\prec \mathrm{B}[M_1, S_2]\succ = \prec \mathrm{B}[RS_2, M_1]\succ$ that the full bisectors $\prec \mathrm{B}[WM_1, WC]\succ$ and $\prec \mathrm{B}[WRS_2, WM_1]\succ$ are transversal along their common slice $M_2 = WM_1$. Since $WC = C$, $WR = U$, and $US_2 = S_2$, the full bisectors $\prec \mathrm{B}[M_2, C]\succ$ and $\prec \mathrm{B}[S_2, M_2]\succ$ are transversal along their common slice M_2.

Lemma 3.1.15 [Appendix, Lemma 4.2.15]. *Let g_1, g_2 be the polar points to the ultraparallel complex geodesics C_1, C_2. A point $x \in \mathbb{B}$ is on the side of the normal vector to the oriented bisector $\prec \mathrm{B}[C_1, C_2]\succ$ if and only if* $\mathrm{Im}\, \dfrac{\langle g_1, x \rangle \langle x, g_2 \rangle}{\langle g_1, g_2 \rangle} \geq 0$.

We require that

$$\mathrm{Im}\, \frac{\langle q_2, h_1 \rangle \langle h_1, m \rangle}{\langle q_2, m \rangle} \geq 0, \qquad \mathrm{Im}\, \frac{\langle Wm, h_1 \rangle \langle h_1, q_2 \rangle}{\langle Wm, q_2 \rangle} \geq 0. \tag{3.1.16}$$

By Lemma 3.1.15, the conditions (3.1.16) express the fact that h_1 belongs to the interior sector at C of the triangle $\Delta(C, M_1, M_2)$.

3.1.17. For brevity, we call the conditions (3.1.2), (3.1.3), (3.1.6), (3.1.9), (3.1.10), (3.1.11), (3.1.14), (3.1.16) the *quadrangle* conditions. Put $s_2 := h_1$ and $q := q_1$.

Proposition 3.1.18. *From the quadrangle conditions, we conclude that both U and W are regular elliptic of order n in PU, that the counterclockwise-oriented transversal triangle $\Delta(S_2, M_2, M_1)$ is transversally adjacent to the counterclockwise-oriented transversal triangle $\Delta(C, M_1, M_2)$, and that both the angle from $\mathrm{B}[C, M_1]$ to $\mathrm{B}[C, M_2]$ at $q \in C$ and the angle from $\mathrm{B}[S_2, M_2]$ to $\mathrm{B}[S_2, M_1]$ at $s_2 \in S_2$ equal $\dfrac{2\pi}{n}$.* ∎

3.2. Examples of disc bundles

From the triangles $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$ constructed in the previous subsection, we obtain a quadrangle $\mathcal{Q}$ whose polyhedron is fibred and constitutes a fundamental region for the group K_n generated by U, W with the defining relations $U^n = W^n = (U^{-1}W)^2 = 1$ in PU. Passing to a subgroup of index n, that is, gluing n copies of the form $W^i\mathcal{Q}$, we arrive at a transversal simple cycle of bisectors which provides a complex hyperbolic disc bundle by Proposition 2.4.9. We calculate the Toledo invariant of the representation provided by the cycle. Using trivializing curves of the triangles, we find how to calculate the Euler number of the polyhedron of the cycle.

Actually, the quadrangle $\mathcal{Q}$ gives rise to a complex hyperbolic disc bundle over the turnover orbifold $\mathbb{S}^2(n, n, 2)$. One can introduce the (rational) Euler number of such bundles as in [Fen]. Since the turnover group K_n possesses a discrete and cocompact action on the Poincaré disc, one can also introduce the Toledo invariant of representations $K_n \to \mathrm{PU}$. In our case, both numbers are easily derivable from those of the cycle glued from copies of $\mathcal{Q}$.

All isometries in this subsection are considered as belonging to SU, unless otherwise stated.

3.2.1. First, we list the facts following from the quadrangle condition 3.1.17 that are not included in Proposition 3.1.18 : The regular elliptic isometry W stabilizes C, fixes the point $q \in C \cap \mathbb{B}$, and

maps M_1 onto M_2. The regular elliptic isometry U stabilizes S_2, fixes the point $s_2 \in S_2 \cap \mathbb{B}$, and maps M_1 onto M_2. The isometry $R := W^{-1}U$ is the reflection in M_1. With a straightforward calculation, we obtain $W^n U^{-n} = \delta$, where $\delta := \exp \dfrac{2(k + l + np)\pi i}{3}$.

Note that the segment of bisector $W \, \mathrm{B}[M_1, S_2]$ is a continuation of the segment $\mathrm{B}[S_2, M_2]$. Indeed, the reflection $R^W := WRW^{-1}$ in M_2 maps S_2 onto $WRW^{-1}S_2 = WU^{-1}S_2 = WS_2$.

3.2.2. Discreteness of K_n. Let $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$ be the triangles constructed in Subsection 3.1 and assume that the quadrangle conditions 3.1.17 are valid. We call the configuration

$$\mathcal{Q} := \big(\mathrm{B}[C, M_1], \mathrm{B}[M_1, S_2], \mathrm{B}[S_2, M_2], \mathrm{B}[M_2, C] \big)$$

the *quadrangle* of bisectors. Since the transversal triangles $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$ are transversally adjacent, the configuration $\mathcal{Q}$ is transversal. By Remark 2.3.6, $\mathcal{Q}$ is simple and the polyhedron Q of $\mathcal{Q}$ is the gluing of the polyhedra of the triangles (see 2.4.4). By Theorem 2.5.2 and Remark 2.4.5, Q is fibred.

Applying [AGr3, Theorem 3.5], we will show that $Q \cap \mathbb{B}$ is a fundamental polyhedron for $K_n \subset \mathrm{PU}$. As in the proof of Theorem 2.2.4, in view of Lemma 2.2.3, we only need to verify the Conditions (1) and (2) of [AGr3, Theorem 3.5]. Since W maps $\mathrm{B}[C, M_1]$ onto $\mathrm{B}[C, M_2]$ and U maps $\mathrm{B}[S_2, M_1]$ onto $\mathrm{B}[S_2, M_2]$, the Condition (1) follows from the definition of counterclockwise-oriented transversal triangles. There are three geometrical cycles of edges. The cycle of C have total angle 2π at $q \in C$ by Proposition 3.1.18. The same concerns the cycle of S_2 and $s_2 \in S_2$. The geometric cycle of M_1 has length 4 due to the relation $U^{-1}WU^{-1}W = 1$. In order to verify that the total angle at a point in $M_1 \cap \mathbb{B}$ is 2π, we recall that W sends $\mathrm{B}[C, M_1]$ onto $\mathrm{B}[C, M_2]$ and that $W \, \mathrm{B}[M_1, S_2]$ is a continuation of $\mathrm{B}[S_2, M_2]$ (see 3.2.1). This implies that the sum of two consecutive angles at (conjugates of) the point in question equals π.

3.2.3. Cycle $\mathcal{C}$, polyhedron P, and disc bundle $M(n, l, k, p)$. Gluing the copies $W^i \mathcal{Q}$, $i = 1, 2$, $\ldots, n$, we obtain a simple transversal cycle of bisectors $\mathcal{C} := (\mathrm{B}_1, \ldots, \mathrm{B}_n)$, where $\mathrm{B}_i := \mathrm{B}[S_i, S_{i+1}]$ and $S_i := W^{i-2}S_2$ (the indices are modulo n). Indeed, S_2 and M_2 are ultraparallel and the reflection in M_2 maps S_2 onto $S_3 = WS_2$ by 3.2.1. Hence, S_2 and S_3 are ultraparallel and M_2 is the middle slice of $\mathrm{B}_2 = \mathrm{B}[S_2, S_3]$. So, $R_i := R^{W^{i-1}}$ is the reflection in the middle slice $M_i := W^{i-2}M_2$ of $\mathrm{B}_i = W^{i-2} \, \mathrm{B}_2$ and

$$R_n \ldots R_1 = W^n(W^{-1}R)^n = W^{n-1}(RW^{-1})^n W = W^{n-1}U^{-n}W = W^n U^{-n} = \delta \qquad (\mathbf{3.2.4})$$

by 3.2.1. Since the triangle $\Delta(S_2, M_2, M_1)$ is transversal, the bisectors $\prec \mathrm{B}_1 \succ$ and $\prec \mathrm{B}_2 \succ$ are transversal along S_2. This implies the transversality of $\mathcal{C}$. The fact that $\Delta'_2 := \Delta(S_2, M_2, M_1)$ is transversally adjacent to $\Delta_2 := \Delta(C, M_1, M_2)$ implies that $\Delta'_i := W^{i-2}\Delta'_2$ is transversally adjacent to $\Delta_i := W^{i-2}\Delta_2$ and that $C = WC$ is a centre of $\mathcal{C}$. By Proposition 3.1.18, the central angle at $q \in C$ is 2π because W fixes q. By Criterion 2.3.9, $\mathcal{C}$ is simple.

Denote by P the polyhedron of the cycle $\mathcal{C}$. Being glued from fibred quadrangles, P is fibred.

The total angle of $\mathcal{C}$ at $s_2 \in S_2 \cap \mathbb{B}$ equals 2π. Indeed, the oriented angle from $\mathrm{B}_2 \succ$ to $\mathrm{B}_1^- \succ$ at $s_2 \in S_2$ equals $\dfrac{2\pi}{n}$ by Proposition 3.1.18. Put $s_i := W^{i-2}s_2$. Then the oriented angle from $\mathrm{B}_i \succ$ to $\mathrm{B}_{i-1}^- \succ$ at $s_i \in S_i$ equals $\dfrac{2\pi}{n}$. It remains to observe that $R_i s_i = W^{i-1}RW^{1-i}W^{i-2}s_2 = W^{i-1}U^{-1}s_2 = W^{i-1}s_2 = s_{i+1}$.

By Proposition 2.4.9, we arrive at the complex hyperbolic disc bundle $M(n, l, k, p)$ diffeomorphic to $\mathbb{B}/G_n$ if n is even and to $\mathbb{B}/T_n$ if n is odd, where G_n or T_n is a torsion-free subgroup of index 2 or 4 in H_n (see 2.1.10), hence, of index $2n$ or $4n$ in K_n, respectively.

3.2.5. Toledo invariant of $\mathcal{C}$. As in 2.1.11 and Proposition 2.1.15, we can calculate the Toledo invariant of the representation of H_n (see 2.1.10) provided by the cycle $\mathcal{C}$.

Proposition 3.2.6. *Suppose that the quadrangle conditions 3.1.17 are valid. Let t be an integer such that $0 \leq t < 3n$ and $t \equiv 2np - k - l \mod 3n$. Then the Toledo invariant of the representation of H_n provided by the cycle $\mathcal{C}$ equals $\frac{2}{3}t - n$.*

Proof. The point $s_2 \in S_2$ generates a meridional curve of $\mathcal{C}$ with vertices $s_{i+1} := R_i s_i$, where $R_i := R^{W^{i-1}}$ and $s_{n+1} = \delta s_1$ (see 3.2.4). Following the proof of Proposition 2.1.15, we take $c := q$ and, using (2.1.16), obtain

$$\tau = \frac{1}{\pi} \sum_{i=1}^n \left(\operatorname{Arg} \frac{\langle q, s_{i+1} \rangle}{\langle q, s_i \rangle} - \pi \right).$$

We have

$$s_i = R_{i-1} \ldots R_2 s_2 = W^{i-2}(RW^{-1})^{i-2} s_2 = W^{i-2} U^{2-i} s_2 = u_1^{2(2-i)} W^{i-2} s_2$$

because $s_2 = h_1$ is an eigenvector of U with eigenvalue u_1^2 (see 3.1.17 and 3.1.1). Since q is an eigenvector of W with eigenvalue w_1^2,

$$\langle q, s_i \rangle = u_1^{2i-4} \langle q, W^{i-2} s_2 \rangle = u_1^{2i-4} \langle W^{2-i} q, s_2 \rangle = u_1^{2i-4} w_1^{4-2i} \langle q, s_2 \rangle = (u_1^2 w_1^{-2})^{i-2} \langle q, s_2 \rangle.$$

Hence, $\operatorname{Arg} \dfrac{\langle q, s_{i+1} \rangle}{\langle q, s_i \rangle} = \operatorname{Arg}(u_1^2 w_1^{-2}) = \dfrac{2t\pi}{3n}$ and $\tau = \frac{2}{3}t - n$. ∎

3.2.7. Holonomy of the quadrangle $\mathcal{Q}$. In order to calculate the Euler number of P, we need to find explicitly a trivializing curve of the polyhedron Q. Theorem 2.5.2 provides trivializing curves of $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$. Therefore, we only need to study a contribution coming from the mutual position of the triangles.

Pick a generic point $z \in \partial_\infty C$ and introduce the following curves and points:
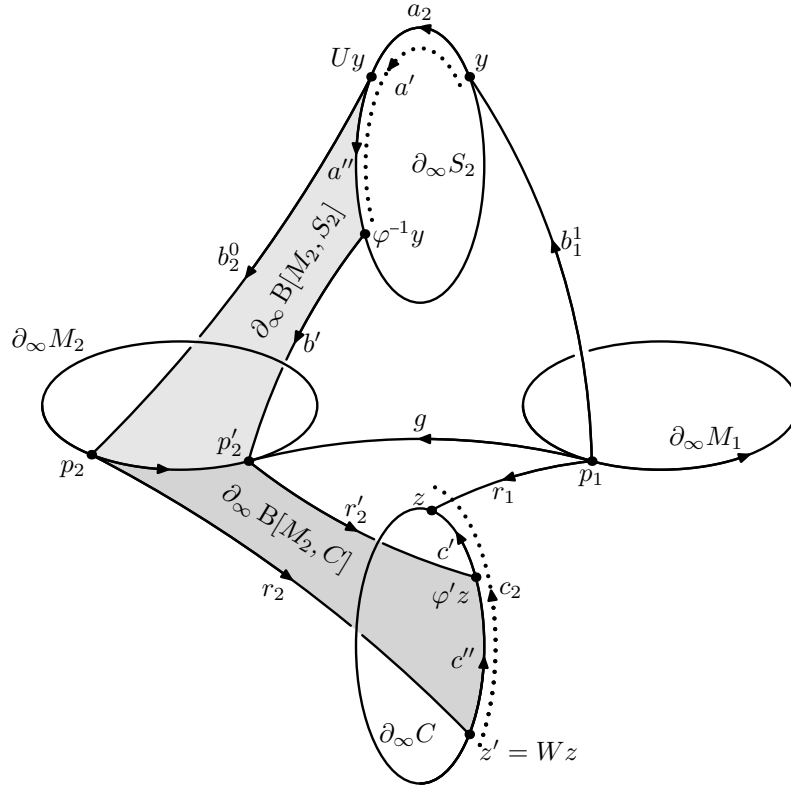
- the meridional[7] curve $r_1^{-1} \subset \partial_\infty \mathrm{B}[C, M_1]$ that begins with z and ends with $p_1 \in \partial_\infty M_1$,
- the meridional curve $b_1^1 \subset \partial_\infty \mathrm{B}[M_1, S_2]$ that begins with p_1 and ends with $y \in \partial_\infty S_2$,
- the naturally oriented simple arc $a_2 \subset \partial_\infty S_2$ that begins with y and ends with $Uy \in \partial_\infty S_2$,
- the meridional curve $b_2^0 \subset \partial_\infty \mathrm{B}[S_2, M_2]$ that begins with Uy and ends with $p_2 \in \partial_\infty M_2$,
- the meridional curve $r_2 \subset \partial_\infty \mathrm{B}[M_2, C]$ that begins with p_2 and ends with $z' \in \partial_\infty C$,
- the naturally oriented simple arc $c_2 \subset \partial_\infty C$ that begins with z' and ends with z.

Since Q is fibred, $\partial_\infty Q$ is a solid torus and the group $H_1(\partial_\infty Q, \mathbb{Z})$ is generated by $[c]$, where c stands for the naturally oriented boundary of C. Hence, there exists $f \in \mathbb{Z}$ such that $[d_2] = f[c]$, where the closed curve d_2 is given by

$$d_2 := r_1^{-1} \cup b_1^1 \cup a_2 \cup b_2^0 \cup r_2 \cup c_2.$$

In order to express f in terms of the holonomies φ of $\Delta(S_2, M_2, M_1)$ and φ' of $\Delta(C, M_1, M_2)$, we introduce the following notation. Let S be an oriented circle and let $t_1, t_2, t_3 \in S$ be pairwise distinct. We define $o(t_1, t_2, t_3) = 0$ if t_1, t_2, t_3 are in the cyclic order of S and $o(t_1, t_2, t_3) = 1$, otherwise. For an isometry I of a complex geodesic E and for $e \in \partial_\infty E$, we put $\lambda(e, I) = 0$ if e belongs to the L-part of $\partial_\infty E$ with respect to I and $\lambda(e, I) = 1$, otherwise.

Lemma 3.2.8. $f = \lambda(y, \varphi) + \lambda(z, \varphi') + o(y, Uy, \varphi^{-1}y) - o(z, Wz, \varphi'z)$.

Proof. First, let us assume that z belongs to the L-part of $\partial_\infty C$ and y, to the L-part of $\partial_\infty S_2$. We introduce a few more curves. The meridional curve $g \subset \partial_\infty \mathrm{B}[M_1, M_2]$ begins with p_1 and ends with some $p_2' \in \partial_\infty M_2$. By definition, there exists a unique meridional curve $b' \subset \partial_\infty \mathrm{B}[S_2, M_2]$ such that $b' \cup g^{-1} \cup b_1^1 \cup a'$ is a trivializing curve of $\Delta(S_2, M_2, M_1)$, where $a' \subset \partial_\infty S_2$ is a simple arc following the natural orientation of $\partial_\infty S_2$. Clearly, b' begins with $\varphi^{-1}y$ and ends with p_2'. Similarly, there exists a unique meridional curve $r_2' \subset \partial_\infty \mathrm{B}[M_2, C]$ such that $r_1^{-1} \cup g \cup r_2' \cup c'$ is a trivializing curve of $\Delta(C, M_1, M_2)$, where $c' \subset \partial_\infty C$ is a simple arc drawn from $\varphi'z$ to z following the natural orientation of $\partial_\infty C$. So, r_2' ends with $\varphi'z$.

By Remark 2.4.5, $r_1^{-1} \cup b_1^1 \cup a' \cup b' \cup r_2' \cup c'$ is a trivializing curve of Q. In terms of 1-chains modulo boundaries, that is, in $C_1(\partial_\infty Q, \mathbb{Z})/\partial C_0(\partial_\infty Q, \mathbb{Z})$, this fact can be written as $-[r_1] + [b_1^1] + [a'] + [b'] + [r_2'] + [c'] = 0$.

Denote by $a'' \subset \partial_\infty S_2$ the simple arc from Uy to $\varphi^{-1}y$ that follows the natural orientation of $\partial_\infty S_2$. It is easy to see that $[a_2] + [a''] - [a'] = o(y, Uy, \varphi^{-1}y)[\partial_\infty S_2]$ in terms of 1-chains modulo boundaries. Let $c'' \subset \partial_\infty C$ stand for the simple arc from z' to $\varphi'z$ that follows the natural orientation of $\partial_\infty C$. As above, $[c''] + [c'] - [c_2] = o(z, z', \varphi'z)[c]$. Since $-[r_2] - [b_2^0] + [a''] + [b'] + [r_2'] - [c''] = 0$, $[\partial_\infty S_2] = [c]$, and $[d_2] = -[r_1] + [b_1^1] + [a_2] + [b_2^0] + [r_2] + [c_2]$, we have

$$[d_2] = -[r_1] + [b_1^1] + [a_2] + [b_2^0] + [r_2] + [c_2] - [r_2] - [b_2^0] + [a''] + [b'] + [r_2'] - [c''] +$$

$$+ [r_1] - [b_1^1] - [a'] - [b'] - [r_2'] - [c'] = [a_2] + [a''] - [a'] + [c_2] - [c''] - [c'] = \big(o(y, Uy, \varphi^{-1}y) - o(z, z', \varphi'z)\big)[c].$$

We claim that $z' = Wz$. Indeed, U maps $\mathrm{B}[S_2, M_1]$ onto $\mathrm{B}[S_2, M_2]$ and y to Uy. Hence, it maps b_1^1 onto b_2^0. In particular, $p_2 = Up_1$. It follows from $Rp_1 = p_1$ and $U = WR$ that $Up_1 = Wp_1$. So, $p_2 = Wp_1$. Now, by similar arguments, W maps r_1 onto r_2, implying $z' = Wz$.

When (say) y belongs to the R-part of $\partial_\infty S_2$, one can readily understand that the required correction equals $\lambda(y, \varphi)$. ∎

Clearly, the terms $\lambda(y, \varphi)$ and $\lambda(z, \varphi')$ vanish if the triangles $\Delta(S_2, M_2, M_1)$ and $\Delta(C, M_1, M_2)$ are both elliptic.

3.2.9. Euler number. In order to find the Euler number of P, we need a few auxiliary facts.

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[7]We denote by x^{-1} the (not necessarily closed) curve x taken with the opposite orientation.

Put $U_i := U^{W^{i-2}}$ and $s_i := W^{i-2}s_2$. The equalities $U_2 S_2 = S_2$ and $U_2 s_2 = s_2$ imply $U_i S_i = S_i$ and $U_i s_i = s_i$ by 3.2.1.

Lemma 3.2.10. $R_i U_i^j = U_{i+1}^j R_i$.

Proof. It suffices to show that $R_i U_i = U_{i+1} R_i$. The relations $U = WR$ and $R^2 = 1$ imply the relations $RW^{-1}U = 1$ and $UR = W$. Since $R_i = R^{W^{i-1}}$,

$$R_i U_i = W^{i-1} R W^{1-i} W^{i-2} U W^{2-i} = W^{i-1} R W^{-1} U W^{2-i} = W,$$

$$U_{i+1} R_i = W^{i-1} U W^{1-i} W^{i-1} R W^{1-i} = W^{i-1} U R W^{1-i} = W. \qquad \blacksquare$$

Lemma 3.2.11. *In the complex geodesic C, the isometry W^{-1} is the rotation about $q \in C$ by the angle $\beta := \dfrac{2(l+1)\pi}{n}$. In the complex geodesic S_i, the isometry U_i is the rotation about $s_i \in S_i$ by the angle $\alpha := \dfrac{2(k+1)\pi}{n}$.*

Proof. In the basis q, q_2, q_3 of eigenvectors of W, the points in $C \cap \overline{\mathbb{B}}$ have the form $\begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix}$, where $z \in \mathbb{C}$ and $|z| \le 1$. In this way, we identify C with the closed unit disc in $\mathbb{C}$ centred at 0 so that q corresponds to 0. Hence, in terms of z, W^{-1} acts as the multiplication by $w_1^2 w_3^{-2}$, implying the first assertion. The same arguments work for $U_2 = U$ and $s_2 \in S_2$, which implies the second assertion. $\blacksquare$

Definition 3.2.12. Let $a \subset \partial_\infty S_i$ be an arc (not necessarily simple) that begins with y and ends with $x = U_i^j y$, $j \in \mathbb{Z}$. We call such an arc *integer*. Assign to an integer arc $a \subset \partial_\infty S_i$ its *angular measure* ℓa (with respect to the centre s_i) by the following rules:

• For $j = 1$ and a simple integer arc $a \subset \partial_\infty S_i$ drawn in the counterclockwise direction, we put $\ell a := \alpha$ (see Lemma 3.2.11).
• If $a \subset \partial_\infty S_i$ is integer, then $\ell a^{-1} = -\ell a$.
• If $a = a' \cup a''$ with integer $a', a'' \subset \partial_\infty S_i$, then $\ell a = \ell a' + \ell a''$.
• Two integer arcs that are homotopic in $\partial_\infty S_i$ have the same angular measure.

Lemma 3.2.13. *Let $a \subset \partial_\infty S_i$ be an integer arc. Then the arc $R_i a \subset \partial_\infty S_{i+1}$ is integer and $\ell R_i a = \ell a$.*

Proof. The arc a begins with y and ends with $x = U_i^j y$, $j \in \mathbb{Z}$. Hence, the arc $R_i a \subset \partial_\infty S_{i+1}$ begins with $R_i y$ and ends with $R_i x = R_i U_i^j y = U_{i+1}^j R_i y$ by Lemma 3.2.10. It remains to observe that the fact is valid for $j = 1$ and any simple integer arc a drawn in the counterclockwise direction. $\blacksquare$

Proposition 3.2.14. *Under the quadrangle conditions 3.1.17, the Euler number of P equals $nf - k - l - 2$, where f is calculated in Lemma 3.2.8.*

Proof. We introduce the following curves and points (see also 3.2.7) :

$$a_i := W^{i-2}a_2 \subset \partial_\infty S_i, \qquad b_i^0 := W^{i-2}b_2^0 \subset \partial_\infty \mathrm{B}[S_i, M_i], \qquad b_i^1 := W^{i-1}b_1^1 \subset \partial_\infty \mathrm{B}[M_i, S_{i+1}],$$

$$c_i := W^{i-2}c_2 \subset \partial_\infty C, \qquad d_i := W^{i-2}d_2 \subset W^{i-2}\partial_\infty Q, \qquad r_i := W^{i-1}r_1 \subset \partial_\infty \mathrm{B}[M_i, C].$$

The facts that $W p_1 = p_2$, $W\,\mathrm{B}[M_1, C] = \mathrm{B}[M_2, C]$, and r_2 is the meridional curve of $\mathrm{B}[M_2, C]$ generated by p_2 imply that $W r_1 = r_2$. So, the definition of the r_i's is correct. By the same reason, we can define $p_i = W^{i-1}p_1 \in \partial_\infty M_i$. Note that b_i^0 ends with p_i and b_i^1 begins with p_i. Since $b_i^0 \subset \mathrm{B}[S_i, M_i]$ and $b_i^1 \subset \mathrm{B}[M_i, S_{i+1}]$ are meridional curves, they form the meridional curve $b_i := b_i^0 \cup b_i^1 \subset \mathrm{B}[S_i, S_{i+1}]$. We obtain the closed curve

$$\gamma := a_1 \cup b_1 \cup \cdots \cup a_n \cup b_n \subset T,$$

where $T := \partial\partial_\infty P$ is the torus of P (see Definition 2.2.2).

Since P is fibred, $\partial_\infty P$ is a solid torus and the group $H_1(\partial_\infty P, \mathbb{Z})$ is generated by $[s]$, where s stands for the naturally oriented boundary of an arbitrary slice of $\mathcal{C}$. Note also that $[s] = [c]$.

The fact that $WP = P$ and $WC = C$ implies $W[c] = [c]$. Therefore, $[d_1 \cup \cdots \cup d_n] = [d_1] + \cdots + [d_n] = nf[c]$. On the other hand, $[d_1 \cup \cdots \cup d_n] = [\gamma] + [c_1 \cup \cdots \cup c_n]$ and $[c_1 \cup \cdots \cup c_n] = (l+1)[c]$ by Lemma 3.2.11 because the angular measure of every c_i with respect to the centre q equals β. Thus, $[\gamma] = (nf - l - 1)[c]$.

Denote by b the meridional curve of $\mathcal{C}$ (see 2.1.8) that includes b_1. We have $b = b'_1 \cup \cdots \cup b'_n$, where b'_i is a meridional curve of $\mathrm{B}[S_i, S_{i+1}]$ and $b'_1 = b_1$. Let $a'_1 \subset \partial_\infty S_1$ be the initial point of b_1 and, by induction, let $a'_{i+1} = R_i a'_i \cup a_{i+1}$ for $i = 1, \ldots, n$. Using Lemma 2.1.4, it is easy to see that the curves $a'_i \cup b_i$ and $b'_i \cup R_i a'_i$ are well defined and homotopic in the cylinder $\partial_\infty \mathrm{B}[S_i, S_{i+1}]$. In terms of 1-chains modulo boundaries in the solid torus $\partial_\infty P$, this implies that $[a'_i] + [b_i] = [b'_i] + [R_i a'_i]$ for all i. Lemma 3.2.13 provides the equality $\ell a'_i = (i - 1)\alpha$ (see also Lemma 3.2.11). Indeed, $W = UR$ and $a_{i+1} = Wa_i$ imply $W = U_{i+1}R_i$ and $a_{i+1} = U_{i+1}R_i a_i$. The equality $\ell a_i = \alpha$ follows from the fact that R_i and U_{i+1} preserve the corresponding angular measures.

By induction, γ is homotopic in T to $b \cup a'_{n+1}$. Hence, $(nf - l - 1)[c] = [\gamma] = [b] + (k+1)[c] = (e_P + k + 1)[c]$. ∎
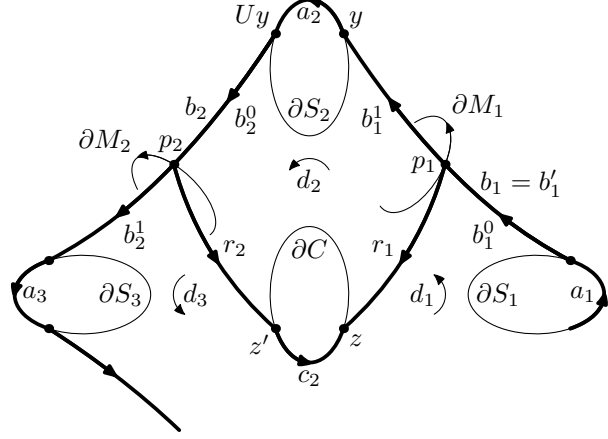
3.2.15. Propositions 2.4.9, 3.2.14, and 3.2.6 have the following

Corollary 3.2.16. *Under the quadrangle conditions* 3.1.17, *the Euler number* e *of the disc bundle* $M(n, l, k, p)$ *over an orientable closed surface* Σ *equals* $4(nf - k - l - 2)$ *if* n *is odd and* $2(nf - k - l - 2)$ *if* n *is even. Moreover,* $2(\chi + e) \equiv 3\tau \mod 8n$ *for odd* n *and* $2(\chi + e) \equiv 3\tau \mod 4n$ *for even* n, *where* χ *is the Euler characteristic of* Σ *and* τ *stands for the Toledo invariant of the representation of* $\pi_1 \Sigma$ *associated to* $M(n, l, k, p)$. ∎

Remark 3.2.17. *The rational Euler number of the disc bundle over the turnover orbifold* $\mathbb{S}^2(n, n, 2)$ *provided by the quadrangle equals* $f - \frac{k+l+2}{n}$. *The corresponding Toledo invariant is* $\frac{2t}{3n} - 1$ *(see Proposition* 3.2.6 *for the definition of* t*).* ∎

3.3. Some interesting examples.

In this subsection, we list some explicit examples of complex hyperbolic disc bundles of the type $M(n, l, k, p)$ obtained with straightforward computer calculations. All we need is to find parameters n, l, k, p satisfying the quadrangle conditions 3.1.17, a task easily achievable with the help of any computational tool. Our particular program has a wide margin of error, thus guaranteeing the validity of our results in the sense that we definitely prefer to lose an existing example rather than to get a doubtful one. In what follows, e, g, χ, and τ stand respectively for the Euler number of the disc bundle $M(n, l, k, p) \to \Sigma$, the genus of Σ, the Euler characteristic of Σ, and the Toledo invariant of the corresponding representation $\pi_1 \Sigma \to \mathrm{PU}$.

We have tested all $n \leq 1001$ and, for every n, all possible values of l, k, p. The $\mathbb{C}$-Fuchsian examples have been discarded. The results are as follows:

• There is no example for $n < 9$ or for $n = 11, 12$. For any other $n \leq 1001$, there exists at least one example. The total number of examples is 308359. There are exactly 89546 examples with integer

Toledo invariant. Therefore, there exist discrete and faithful representations $\pi_1\Sigma \to \mathrm{PU}$ with fractional Toledo invariant.

• For every example, both triangles $\Delta(C, M_1, M_2)$ and $\Delta(S_2, M_2, M_1)$ are elliptic, $o(z, Wz, \varphi'z) = 0$, $f = o(y, Uy, \varphi^{-1}y)$, and $p + f = 2$. The last equality can be seen as some tiny evidence supporting the complex hyperbolic variant of the GLT-conjecture (see below).

• For every Σ with $\chi\Sigma < 0$ and for an arbitrary even integer τ subject to the Toledo necessary condition $|\tau| \leq |\chi|$, it is constructed in [GKL] a complex hyperbolic disc bundle. Therefore, each of our examples with integer τ provides a couple of nonhomeomorphic complex hyperbolic disc bundles over the same Σ and with the same τ. This implies that there exist discrete and faithful representations $\pi_1\Sigma \to \mathrm{PU}(2,1)$ that lie in the same connected component of the space of representations [Xia] but not in the same connected component of the space of discrete and faithful representations.

• Every example satisfies the inequalities $\tau < 0$ and $\frac{1}{2}\chi < e < 0$. All previously known examples, including those constructed in [GKL], satisfy the inequalities $\chi \leq e \leq \frac{1}{2}\chi$.

• Every example satisfies the equality $2(\chi + e) = 3\tau$. This equality is a necessary condition for the existence of a holomorphic section of the bundle. It suggests the following conjecture: Every (or at least one) $M(n, l, k, p)$ possesses a disc bundle structure admitting a holomorphic section.

• The following table contains all examples with extreme values of g, e, and e/χ :

Manifold	g	χ	e	τ	Comment
$M(10, 6, 3, 1)$	4	-6	-2	$-5\frac{1}{3}$	minimal g, maximal e
$M(9, 4, 4, 1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(9, 5, 3, 1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(9, 6, 2, 1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(14, 7, 7, 1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(14, 8, 6, 1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(14, 9, 5, 1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(14, 10, 4, 1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(14, 11, 3, 1)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(14, 0, 0, 2)$	6	-10	-4	$-9\frac{1}{3}$	next to minimal g, next to maximal e
$M(16, 0, 0, 2)$	7	-12	-4	$-10\frac{2}{3}$	next to maximal e
$M(18, 0, 0, 2)$	8	-14	-4	-12	next to maximal e
$M(20, 0, 0, 2)$	9	-16	-4	$-13\frac{1}{3}$	next to maximal e
$M(22, 0, 0, 2)$	10	-18	-4	$-14\frac{2}{3}$	next to maximal e
$M(24, 0, 0, 2)$	11	-20	-4	-16	next to maximal e
$M(26, 0, 0, 2)$	12	-22	-4	$-17\frac{1}{3}$	next to maximal e
$M(28, 0, 0, 2)$	13	-24	-4	$-18\frac{2}{3}$	minimal $e/\chi = \frac{1}{6}$, next to maximal e

In principle, some examples in the above table could be isometric, for instance, those with $n = 14$. However, this is not the case as it is easy to show that these bundles provide distinct points in the space $\mathcal{T}_n$ of discrete and faithful representations $\varrho : \pi_1\Sigma \to \mathrm{PU}$ modulo conjugation by PU. Note that all examples with $n = 14$ have the same known discrete invariants. We believe that those with $p = 1$ are in the same connected component of $\mathcal{T}_{14}$. Moreover, we expect that there exist new Toledo-like discrete invariants that distinguish the cases $p = 1$ and $p = 2$ thus showing that the corresponding manifolds provide points in different connected components of $\mathcal{T}_{14}$. Such a behaviour should not be specific for $n = 14$.

The fact that the equivalences in Corollary 3.2.16 turn out to be equalities in our explicit examples is

a sort of evidence in support of the complex hyperbolic GLT-conjecture analogous to the real hyperbolic one stated in [GLT] : A disc bundle over a closed orientable surface admits complex hyperbolic structure if and only if $|e| \leq |\chi|$ and $\chi < 0$.

• The two tables below display all 55 examples that satisfy the inequality $\frac{1}{3}\chi \leq e$ and, therefore, due to [Kui], admit a real hyperbolic structure:

Manifold	g	χ	e	τ
$M(10,6,3,1)$	4	-6	-2	$-5\frac{1}{3}$
$M(16,0,0,2)$	7	-12	-4	$-10\frac{2}{3}$
$M(18,0,0,2)$	8	-14	-4	-12
$M(20,0,0,2)$	9	-16	-4	$-13\frac{1}{3}$
$M(22,0,0,2)$	10	-18	-4	$-14\frac{2}{3}$
$M(22,1,0,2)$	10	-18	-6	-16
$M(24,0,0,2)$	11	-20	-4	-16
$M(24,1,0,2)$	11	-20	-6	$-17\frac{1}{3}$
$M(26,0,0,2)$	12	-22	-4	$-17\frac{1}{3}$
$M(26,1,0,2)$	12	-22	-6	$-18\frac{2}{3}$
$M(28,0,0,2)$	13	-24	-4	$-18\frac{2}{3}$
$M(28,1,0,2)$	13	-24	-6	-20
$M(28,1,1,2)$	13	-24	-8	$-21\frac{1}{3}$
$M(28,2,0,2)$	13	-24	-8	$-21\frac{1}{3}$
$M(17,0,0,2)$	14	-26	-8	$-22\frac{2}{3}$
$M(30,1,1,2)$	14	-26	-8	$-22\frac{2}{3}$
$M(30,2,0,2)$	14	-26	-8	$-22\frac{2}{3}$
$M(32,1,1,2)$	15	-28	-8	-24
$M(32,2,0,2)$	15	-28	-8	-24
$M(19,0,0,2)$	16	-30	-8	$-25\frac{1}{3}$
$M(34,1,1,2)$	16	-30	-8	$-25\frac{1}{3}$
$M(34,2,1,2)$	16	-30	-10	$-26\frac{2}{3}$
$M(34,3,0,2)$	16	-30	-10	$-26\frac{2}{3}$
$M(36,1,1,2)$	17	-32	-8	$-26\frac{2}{3}$
$M(36,2,1,2)$	17	-32	-10	-28
$M(21,0,0,2)$	18	-34	-8	-28
$M(38,2,1,2)$	18	-34	-10	$-29\frac{1}{3}$
$M(40,2,2,2)$	19	-36	-12	-32

Manifold	g	χ	e	τ
$M(40,3,1,2)$	19	-36	-12	-32
$M(23,0,0,2)$	20	-38	-8	$-30\frac{2}{3}$
$M(23,1,0,2)$	20	-38	-12	$-33\frac{1}{3}$
$M(42,2,2,2)$	20	-38	-12	$-33\frac{1}{3}$
$M(44,2,2,2)$	21	-40	-12	$-34\frac{2}{3}$
$M(25,0,0,2)$	22	-42	-8	$-33\frac{1}{3}$
$M(25,1,0,2)$	22	-42	-12	-36
$M(46,3,2,2)$	22	-42	-14	$-37\frac{1}{3}$
$M(27,0,0,2)$	24	-46	-8	-36
$M(27,1,0,2)$	24	-46	-12	$-38\frac{2}{3}$
$M(52,3,3,2)$	25	-48	-16	$-42\frac{2}{3}$
$M(29,1,0,2)$	26	-50	-12	$-41\frac{1}{3}$
$M(29,1,1,2)$	26	-50	-16	-44
$M(29,2,0,2)$	26	-50	-16	-44
$M(31,1,1,2)$	28	-54	-16	$-46\frac{2}{3}$
$M(31,2,0,2)$	28	-54	-16	$-46\frac{2}{3}$
$M(33,1,1,2)$	30	-58	-16	$-49\frac{1}{3}$
$M(35,1,1,2)$	32	-62	-16	-52
$M(35,2,1,2)$	32	-62	-20	$-54\frac{2}{3}$
$M(37,1,1,2)$	34	-66	-16	$-54\frac{2}{3}$
$M(37,2,1,2)$	34	-66	-20	$-57\frac{1}{3}$
$M(41,2,2,2)$	38	-74	-24	$-65\frac{1}{3}$
$M(41,3,1,2)$	38	-74	-24	$-65\frac{1}{3}$
$M(43,2,2,2)$	40	-78	-24	-68
$M(45,2,2,2)$	42	-82	-24	$-70\frac{2}{3}$
$M(47,3,2,2)$	44	-86	-28	-76
$M(53,3,3,2)$	50	-98	-32	$-86\frac{2}{3}$

• The following tables show all examples for $n = 101$ ($g = 98$, $\chi = -194$). They illustrate a typical behaviour of l, k, p, and the Euler number for some fixed n :

$e = -96,\ \tau = -193\frac{1}{3}$	$e = -96,\ \tau = -193\frac{1}{3}$
$M(101, 62, 61, 1)$	$M(101, 79, 44, 1)$
$M(101, 63, 60, 1)$	$M(101, 80, 43, 1)$
$M(101, 64, 59, 1)$	$M(101, 81, 42, 1)$
$M(101, 65, 58, 1)$	$M(101, 82, 41, 1)$
$M(101, 66, 57, 1)$	$M(101, 83, 40, 1)$
$M(101, 67, 56, 1)$	$M(101, 84, 39, 1)$
$M(101, 68, 55, 1)$	$M(101, 85, 38, 1)$
$M(101, 69, 54, 1)$	$M(101, 86, 37, 1)$
$M(101, 70, 53, 1)$	$M(101, 87, 36, 1)$
$M(101, 71, 52, 1)$	$M(101, 88, 35, 1)$
$M(101, 72, 51, 1)$	$M(101, 89, 34, 1)$
$M(101, 73, 50, 1)$	$M(101, 90, 33, 1)$
$M(101, 74, 49, 1)$	$M(101, 91, 32, 1)$
$M(101, 75, 48, 1)$	$M(101, 92, 31, 1)$
$M(101, 76, 47, 1)$	$M(101, 93, 30, 1)$
$M(101, 77, 46, 1)$	$M(101, 94, 29, 1)$
$M(101, 78, 45, 1)$	$M(101, 95, 28, 1)$

$e = -96,\ \tau = -193\frac{1}{3}$	$e = -92,\ \tau = -190\frac{2}{3}$
$M(101, 96, 27, 1)$	$M(101, 96, 26, 1)$
$M(101, 97, 26, 1)$	$M(101, 97, 25, 1)$
$M(101, 98, 25, 1)$	$M(101, 98, 24, 1)$
$M(101, 11, 11, 2)$	$M(101, 11, 10, 2)$
$M(101, 12, 10, 2)$	$M(101, 12, 9, 2)$
$M(101, 13, 9, 2)$	$M(101, 13, 8, 2)$
$M(101, 14, 8, 2)$	$M(101, 14, 7, 2)$
$M(101, 15, 7, 2)$	$M(101, 15, 6, 2)$
$M(101, 16, 6, 2)$	$M(101, 16, 5, 2)$
$M(101, 17, 5, 2)$	$M(101, 17, 4, 2)$
$M(101, 18, 4, 2)$	$M(101, 18, 3, 2)$
$M(101, 19, 3, 2)$	$M(101, 19, 2, 2)$
$M(101, 20, 2, 2)$	$M(101, 20, 1, 2)$
$M(101, 21, 1, 2)$	$M(101, 21, 0, 2)$
$M(101, 22, 0, 2)$	

$e = -88,\ \tau = -188$
$M(101, 10, 10, 2)$
$M(101, 11, 9, 2)$
$M(101, 12, 8, 2)$
$M(101, 13, 7, 2)$
$M(101, 14, 6, 2)$
$M(101, 15, 5, 2)$
$M(101, 16, 4, 2)$

$e = -84,\ \tau = -185\frac{1}{3}$
$M(101, 10, 9, 2)$
$M(101, 11, 8, 2)$
$M(101, 12, 7, 2)$

$e = -80,\ \tau = -182\frac{2}{3}$
$M(101, 9, 9, 2)$

4. Appendix: technical tools, elementary algebraic and geometric background

This section contains proofs of all facts that were just stated in the previous sections. The core of the method we use here is a coordinate-free approach in complex hyperbolic geometry. It originates mainly from the works [San] and [HSa]. With this approach, we significantly reduce the amount of calculations involved in the proofs. It happened to be convenient to work in a slightly extended generality, using extensively the positive part of $\mathbb{P}$. This either does not alter or indeed simplifies the proofs. It is worthwhile mentioning that the pseudo-Riemannian geometry of the positive part is important *per se,* as it is nothing but the geometry of complex geodesics; it will be studied more systematically in [AGr1].

Behind several results of this section there is an important and curious geometrical interpretation that we plan to discuss in other articles. It is related to some kind of parallel displacement. (The explicit formulae can be found in [AGr1] and [AGr2].) This concerns the slice identification, the angle between cotranchal bisectors, and the Kähler potential, for instance.

Sometimes, we prove well-known results. In these cases, the direct references do not simplify the exposition since, usually, we need something developed in the proof.

For the convenience of the reader unfamiliar with complex hyperbolic geometry, we have collected in Subsection 4.1 a series of simple well-known facts. Most of them can be found in [Gol] or [San]. Some of the notation and definitions introduced in this subsection differ from the standard ones.

4.1. Preliminaries: hermitian metric, complex geodesics, geodesics, $\mathbb{R}$-planes, and bisectors

4.1.1. Fix, once and for all, a three-dimensional $\mathbb{C}$-vector space V equipped with an hermitian form $\langle -, - \rangle$ of signature $+ + -$. (In order to deal with the Poincaré disc, one can take $\dim_{\mathbb{C}} V = 2$ and signature $+-$.) Depending on the context, we will use the elements of V to denote the points in the complex projectivization $\mathbb{P}$ of V. In general, $\mathbb{P}W$ denotes the complex projectivization of $W \subset V$.

As in 2.1.1, we define

$$\mathbb{B} := \{ p \in \mathbb{P} \mid \langle p, p \rangle < 0 \}, \qquad \partial_\infty \mathbb{B} := \{ p \in \mathbb{P} \mid \langle p, p \rangle = 0 \}, \qquad \overline{\mathbb{B}} := \mathbb{B} \cup \partial_\infty \mathbb{B}.$$

We recall that every linear map $\varphi \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}p, V)$ can be regarded as a tangent vector $t_\varphi \in \mathrm{T}_p \mathbb{P}$ by defining

$$t_\varphi f := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\big(p + \varepsilon \varphi(p)\big)$$

for a local smooth function f on $\mathbb{P}$ and its lift $\hat{f}$ to V. Indeed, for small $\varepsilon \in \mathbb{R}$, we have a smooth curve $c(\varepsilon) \in \mathbb{P}$ whose lift to V is given by $c_0(\varepsilon) := p + \varepsilon \varphi(p)$. The above $t_\varphi f$ is nothing but $\dot{c}(0)f$. In this way,

$$\mathrm{T}_p \mathbb{P} \simeq \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}p, V/\mathbb{C}p).$$

Let p be nonisotropic, i.e., $p \notin \partial_\infty \mathbb{B}$. We have the orthogonal decomposition

$$V = \mathbb{C}p \oplus p^\perp, \qquad v = \pi'[p]v + \pi[p]v,$$

where the orthogonal projections

$$\pi'[p]v := \frac{\langle v, p \rangle}{\langle p, p \rangle} p \in \mathbb{C}p, \qquad \pi[p]v := v - \frac{\langle v, p \rangle}{\langle p, p \rangle} p \in p^\perp$$

do not depend on the choice of a representative $p \in V$. The natural identification

$$\mathrm{T}_p \mathbb{P} \simeq \langle -, p \rangle p^\perp$$

provides a nondegenerate hermitian form on the tangent space $\mathrm{T}_p \mathbb{P}$:

$$\langle v_p, w_p \rangle := -\langle p, p \rangle \langle v, w \rangle, \tag{4.1.2}$$

where $v, w \in p^\perp$ and, for a chosen representative $p \in V$,

$$v_p := \langle -, p \rangle v \in \mathrm{T}_p \mathbb{P}. \tag{4.1.3}$$

Obviously, this hermitian form depends smoothly on a nonisotropic p.

Lemma 4.1.4. *Let $c : [a, b] \to \mathbb{P}$ be a smooth curve and let $c_0 : [a, b] \to V$ be a smooth lift of c. If $c(t_0) \notin \partial_\infty \mathbb{B}$, then $\dot{c}(t_0) = \langle -, c_0(t_0) \rangle \dfrac{\pi\big[c(t_0)\big] \dot{c}_0(t_0)}{\langle c_0(t_0), c_0(t_0) \rangle}$ is the tangent vector to c at $c(t_0)$.*

Proof. Denoting $k := \dfrac{\langle \dot{c}_0(t_0), c_0(t_0) \rangle}{\langle c_0(t_0), c_0(t_0) \rangle}$, we have

$$\dot{c}(t_0)f = \frac{d}{dt}\Big|_{t=t_0} \hat{f}\big(c_0(t)\big) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\big(c_0(t_0) + \varepsilon \dot{c}_0(t_0)\big) =$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\Big(c_0(t_0) + \varepsilon\pi'\big[c(t_0)\big]\dot{c}_0(t_0) + \varepsilon\pi\big[c(t_0)\big]\dot{c}_0(t_0)\Big) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\Big((1 + \varepsilon k)c_0(t_0) + \varepsilon\pi\big[c(t_0)\big]\dot{c}_0(t_0)\Big) =$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\Big(c_0(t_0) + \frac{\varepsilon}{1 + \varepsilon k}\pi\big[c(t_0)\big]\dot{c}_0(t_0)\Big) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\Big(c_0(t_0) + \varepsilon\pi\big[c(t_0)\big]\dot{c}_0(t_0)\Big). \qquad \blacksquare$$

4.1.5. Tance. For $p_1, p_2 \in \mathbb{P}$, we define

$$\mathrm{ta}(p_1, p_2) := \frac{\langle p_1, p_2\rangle\langle p_2, p_1\rangle}{\langle p_1, p_1\rangle\langle p_2, p_2\rangle}.$$

When one of p_1, p_2 is isotropic, our convention is that $\mathrm{ta}(p_1, p_2) := +\infty$ if $\langle p_1, p_2\rangle \neq 0$ and $\mathrm{ta}(p_1, p_2) := 1$ if $\langle p_1, p_2\rangle = 0$. The concept of tance can be used for expressing distances (see, for example, Corollary 4.1.18 and Lemma 4.1.8).

4.1.6. Polar points to projective lines. For every projective line $L \subset \mathbb{P}$, there exists a unique point $p \in \mathbb{P}$ such that $L = \mathbb{P}p^{\perp}$. We call p the *polar* point to L. By definition, the *signature* of L is simply that of the hermitian form on the $\mathbb{C}$-vector subspace in V corresponding to L. The signature of $\mathbb{P}p^{\perp}$ is respectively $++$, $+-$, $+0$ in the cases $p \in \mathbb{B}$, $p \notin \overline{\mathbb{B}}$, $p \in \partial_\infty\mathbb{B}$. Denote by

$$L\,\wr p_1, p_2\wr := \mathbb{P}(\mathbb{C}p_1 + \mathbb{C}p_2)$$

the projective line spanned by two distinct points $p_1, p_2 \in \mathbb{P}$.

Lemma 4.1.7 [Gol, p. 100]. *Let $p_1, p_2 \in \mathbb{P}$ be distinct. Then $\mathbb{P}p_1^{\perp}$ and $\mathbb{P}p_2^{\perp}$ intersect in the polar point p to $L\wr p_1, p_2\wr$. Moreover,*

- $p \in \partial_\infty\mathbb{B}$ *if and only if* $\mathrm{ta}(p_1, p_2) = 1$;
- $p \in \mathbb{B}$ *if and only if* $\mathrm{ta}(p_1, p_2) < 1$ *and* $p_1, p_2 \notin \overline{\mathbb{B}}$.

Proof. The first assertion is obvious. Sylvester's Criterion [KoM, p. 113] and the definition of $\mathrm{ta}(p_1, p_2)$ imply that the signature of $L\wr p_1, p_2\wr$ is $+0$ if and only if $\mathrm{ta}(p_1, p_2) = 1$. This proves the second assertion. Suppose that $p \in \mathbb{B}$. Then $p_1, p_2 \notin \overline{\mathbb{B}}$ because $p_1, p_2 \in p^{\perp}$. Applying Sylvester's Criterion to the Gram matrix of p_1, p_2, p, we infer that $\mathrm{ta}(p_1, p_2) < 1$. Conversely, $\mathrm{ta}(p_1, p_2) < 1$ and $p_1, p_2 \notin \overline{\mathbb{B}}$ imply $p \in \mathbb{B}$ by the same criterion. $\qquad \blacksquare$

The *complex geodesics* we deal with in Sections 1–3 are by definition of the form $\mathbb{P}p^{\perp} \cap \overline{\mathbb{B}}$ with positive p. Two distinct complex geodesics are said to be *ultraparallel* if they do not intersect. By Lemma 4.1.7, this is equivalent to the inequality $\mathrm{ta}(p_1, p_2) > 1$ for their polar points p_1 and p_2.

The tance between a point and a complex geodesic is given in the following

Lemma 4.1.8 [Gol, p. 197, Theorem 6.1.1]. *Let $p \notin \overline{\mathbb{B}}$ and let $q \in \mathbb{B}$. Then $\mathrm{ta}(q, \mathbb{P}p^{\perp} \cap \mathbb{B}) = 1 - \mathrm{ta}(p, q)$.*

Proof. It follows from $\langle\pi[p]q, \pi[p]q\rangle = \langle\pi[p]q, q\rangle = \langle q, q\rangle\big(1 - \mathrm{ta}(p, q)\big) < 0$ that $\pi[p]q \in \mathbb{P}p^{\perp} \cap \mathbb{B}$ and that $\mathrm{ta}\big(\pi[p]q, q\big) = 1 - \mathrm{ta}(p, q)$. We choose an orthonormal basis e_1, e_2, e_3 in V such that e_1 and e_2 represent p and $\pi[p]q$, respectively. We can assume that $q = ce_1 + e_2$, $|c| < 1$. Every point in $\mathbb{P}p^{\perp} \cap \mathbb{B}$ can be written in the form $p(z) := e_2 + ze_3$, $|z| < 1$. It remains to observe that

$$\mathrm{ta}\big(p(z), q\big) = \frac{1}{\big(-1 + |z|^2\big)\big(|c|^2 - 1\big)} = \frac{1}{\big(1 - |z|^2\big)\big(1 - |c|^2\big)}. \qquad \blacksquare$$

4.1.9. Geodesics and $\mathbb{R}$-planes. Let $S \subset V$ be an $\mathbb{R}$-vector subspace such that the hermitian form is real and nondegenerate on S. It is immediate that $\mathbb{C}S \simeq \mathbb{C}\otimes_{\mathbb{R}}S$. Therefore, $\mathbb{P}S = \mathbb{P}_{\mathbb{R}}S$. If $\dim_{\mathbb{R}}S = 2$, we call

$$G\,S := \mathbb{P}S = \mathbb{P}_{\mathbb{R}}^1 S \simeq \mathbb{S}^1$$

an *extended geodesic* (or simply a *geodesic*) [San]. If $\dim_{\mathbb{R}} S = 3$, we call

$$\mathrm{R}\, S := \mathbb{P}S = \mathbb{P}^2_{\mathbb{R}} S$$

an *extended* $\mathbb{R}$-*plane* (or simply an $\mathbb{R}$-*plane*) [Gol, p. 80]. When writing $\mathrm{G}\, S$ or $\mathrm{R}\, S$ (or even $\mathbb{P}S$) for a geodesic or real plane, we assume that S is an $\mathbb{R}$-vector subspace such that the hermitian form is real and nondegenerate on S. Every geodesic $\mathrm{G}\, S$ spans its projective line $\mathrm{L}\,\mathrm{G}\, S := \mathbb{P}(\mathbb{C}S)$. By definition, the *signature* of $\mathrm{G}\, S$ is that of $\mathrm{L}\,\mathrm{G}\, S$. It coincides with the signature of the (hermitian) form on S and can be $++$ or $+-$.

Remark 4.1.10. *Let* L *be a projective line and let* $p \in \mathrm{L}$ *be nonisotropic. Then there exists a unique* $q \in \mathrm{L}$ *orthogonal to* p, *i.e.,* $\langle p, q \rangle = 0$. *If* G *is a geodesic such that* $p \in \mathrm{G} \subset \mathrm{L}$, *then* $q \in \mathrm{G}$. ∎

Sylvester's Criterion implies the

Remark 4.1.11. *Suppose that* $\mathrm{ta}(g_1, g_2) \neq 0, 1$ *for some* $g_1, g_2 \in \mathbb{P}$. *Then there exists a unique extended geodesic* $\mathrm{G}\, \wr g_1, g_2 \wr$ *containing* g_1 *and* g_2. ∎

When using the notation $\mathrm{G}\, \wr g_1, g_2 \wr$, we assume by default that $\mathrm{ta}(g_1, g_2) \neq 0, 1$.

4.1.12. In order to write down the equation of the geodesic $\mathrm{G}\, \wr g_1, g_2 \wr$, we put

$$b(x, g_1, g_2) := \frac{\langle g_1, x \rangle \langle x, g_2 \rangle}{\langle g_1, g_2 \rangle} - \frac{\langle g_2, x \rangle \langle x, g_1 \rangle}{\langle g_2, g_1 \rangle}.$$

Note that $b(x, g_1, g_2)$ does not depend on the choice of representatives $g_1, g_2 \in V$.

Lemma 4.1.13. *The geodesic* $\mathrm{G}\, \wr g_1, g_2 \wr$ *is given in* $\mathrm{L}\, \wr g_1, g_2 \wr$ *by the equation* $b(x, g_1, g_2) = 0$.

Proof. $\mathrm{G}\, \wr g_1, g_2 \wr = \mathrm{G}\, S$ for a suitable real subspace $S \subset V$. The equality $b(x, g_1, g_2) = 0$ holds for every $x \in S$ since the hermitian form is real on S and we can take $g_1, g_2 \in S$.

Conversely, suppose that $\dfrac{\langle g_1, x \rangle \langle x, g_2 \rangle}{\langle g_1, g_2 \rangle} \in \mathbb{R}$ for some $x \in \mathrm{L}\, \wr g_1, g_2 \wr$. We can assume that $g_1, g_2 \in S$ and that $x = g_1 + c g_2$ for some $c \in \mathbb{C}$. Then $0 \neq \langle g_1, g_2 \rangle \in \mathbb{R}$ and

$$0 \equiv \langle g_1, x \rangle \langle x, g_2 \rangle \equiv c \langle g_1, g_1 \rangle \langle g_2, g_2 \rangle + \overline{c} \langle g_1, g_2 \rangle \langle g_1, g_2 \rangle \quad \mathrm{mod}\ \mathbb{R}.$$

If $c \notin \mathbb{R}$, then $\langle g_1, g_1 \rangle \langle g_2, g_2 \rangle = \langle g_1, g_2 \rangle \langle g_1, g_2 \rangle$. In other words, $\mathrm{ta}(g_1, g_2) = 1$. ∎

4.1.14. Every geodesic G of signature $+-$ possesses exactly two isotropic points $v_1, v_2 \in \mathrm{G}$ called the *vertices* of G. There is a canonical way to describe such a geodesic in terms of its vertices:

Lemma 4.1.15 [Gol, p. 155]. *Let* G *be a geodesic of signature* $+-$, *let* v_1, v_2 *stand for the vertices of* G, *and let* $g \in \mathrm{G}$ *be a point of signature* $\pm$. *Then we can choose representatives* $v_1, v_2, g \in V$ *such that* $g = v_1 + v_2$ *and* $\begin{pmatrix} 0 & \pm\frac{1}{2} \\ \pm\frac{1}{2} & 0 \end{pmatrix}$ *is the Gram matrix of* v_1, v_2. *The formula* $g(t) := t^{-1} v_1 + t v_2$, $t > 0$, *describes all points in* G *of signature* $\pm$. *For all* t, *we have* $\langle g(t), g(t) \rangle = \pm 1$.

Proof. We can always choose representatives $v_1, v_2, g \in V$ providing the required Gram matrix, $g \in \mathbb{R}v_1 + \mathbb{R}v_2$, and $\langle g, g \rangle = \pm 1$. Replacing g by $-g$ if necessary, we obtain $g = t^{-1} v_1 + t v_2$ for a suitable $t > 0$. It remains to take $t^{-1} v_1, t v_2$ for v_1, v_2. ∎

Lemma 4.1.16. *If* $p, g_1, g_2 \in \mathbb{B}$, *then* $\dfrac{\langle g_1, p \rangle \langle p, g_2 \rangle}{\langle g_1, g_2 \rangle}$ *cannot be real nonnegative.*

Proof. It suffices to show that $\mathrm{Re}\, \dfrac{\langle g_1, p \rangle \langle p, g_2 \rangle}{\langle g_1, g_2 \rangle} < 0$ assuming that $g_1 \neq g_2 \neq p$. Taking the negative point $\pi[q]p$ in place of p, where $q \notin \overline{\mathbb{B}}$ denotes the polar point to $\mathrm{L}\, \wr g_1, g_2 \wr$, we can assume that

$p \in \mathrm{L}\wr g_1, g_2\wr$. Choose representatives such that $\langle g_i, g_i \rangle = -1$, $\langle g_1, g_2 \rangle = a > 1$, and $p = g_1 + cg_2$ for some $c \in \mathbb{C}$. Since $\langle p, p \rangle < 0$, we have $2a \operatorname{Re} c < 1 + |c|^2$. Now,

$$\operatorname{Re}\left(\langle g_1, p \rangle \langle p, g_2 \rangle\right) = (a^2 + 1)\operatorname{Re} c - a\left(1 + |c|^2\right) < \frac{(1 - a^2)\left(1 + |c|^2\right)}{2a} < 0. \qquad \blacksquare$$

4.1.17. Length of geodesics. Let $\mathrm{G}[g_1, g_2] \subset \mathbb{B}$ denote the oriented segment of geodesic joining the points $g_1, g_2 \in \mathbb{B}$. By Lemma 4.1.15, we can parameterize a lift of $\mathrm{G}[g_1, g_2]$ to V as $c_0(t) = e^{-t}v_1 + e^t v_2$, $t \in [0, a]$, for some $a \geq 0$. It is easy to see that $\dot{c}_0(t) = -e^{-t}v_1 + e^t v_2$ is orthogonal to $c_0(t)$. By Lemma 4.1.4, $\langle \dot{c}(t), \dot{c}(t) \rangle = 1$. Therefore, $\ell\,\mathrm{G}[g_1, g_2] = \displaystyle\int_0^a \sqrt{\langle \dot{c}(t), \dot{c}(t) \rangle}\, dt = a$. A straightforward calculation shows that $\operatorname{ta}(g_1, g_2) = \left(\dfrac{e^{-a} + e^a}{2}\right)^2$. We arrive at the

Corollary 4.1.18. *For $g_1, g_2 \in \mathbb{B}$, we have $\cosh^2\left(\operatorname{dist}(g_1, g_2)\right) = \operatorname{ta}(g_1, g_2)$.* $\qquad \blacksquare$

Now, we can see that the metric on $\mathbb{B}$ coincides, up to the scale factor of 4, with the one introduced in [Gol, p. 77].

4.1.19. Bisectors. Let $\mathrm{G}\,S$ be a geodesic and let f stand for the polar point to its projective line $\mathrm{L}\,\mathrm{G}\,S$. The projective cone B over $\mathrm{G}\,S$ with vertex f is said to be an *extended bisector* (or simply a *bisector*). The nonisotropic point f is the *focus* of B. Clearly, f is the only singular point in B. The geodesic $\mathrm{G}\,S$ is the *real spine* of B, the projective line $\mathrm{L}\,\mathrm{G}\,S$ is the *complex spine* of B, and the projective line $\mathrm{L}\,\wr g, f\wr$ is the *slice* of B containing $g \in \mathrm{G}\,S$. In this way, we obtain the *slice decomposition* of B : every point in B different from the focus belongs to a unique slice of B. By Remark 4.1.10, the polar point to every slice of B belongs to the real spine of B. The vertices of the real spine of B (if exist) are the *vertices* of B.
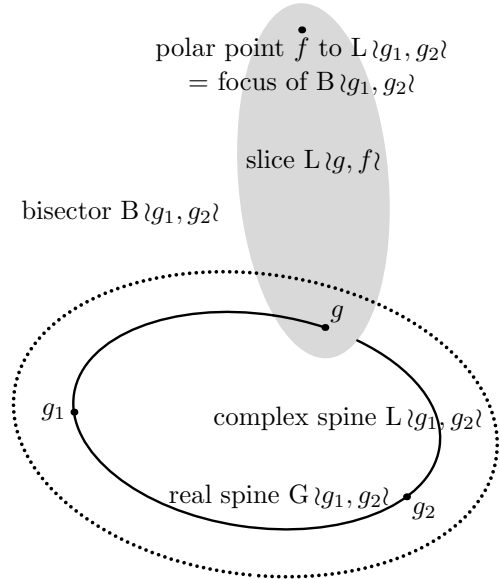
We can also describe the bisector B as $\mathrm{B} = \mathbb{P}W$, where $W := S + \mathbb{C}f$ is a 4-dimensional real subspace in V. This description immediately provides the *meridional decomposition* of B : Let us fix some representative $f \in V$. For every $0 \neq c \in \mathbb{C}$, the hermitian form on $S' := S + \mathbb{R}cf$ is real and nondegenerate. The $\mathbb{R}$-plane $\mathrm{R}\,S' \subset \mathrm{B}$ is said to be a *meridian* of B. Two different meridians intersect exactly in the focus and in the real spine. Obviously, every point in a bisector belongs to some meridian. Such a meridian is unique if the point is different from the focus and does not lie in the real spine.

It is easy to see that the intersection of a meridian of B with the slice $\mathrm{L}\,\wr g, f\wr$ of B is a geodesic (unless g is a vertex of the real spine of B) that contains g.

It is also possible to define a bisector as the hypersurface equidistant (equitant) from two different points. In this article, we do not need such a definition.

Our definition of bisector differs slightly from the common one. A standardly defined bisector is, in our terms, $\mathrm{B} \cap \overline{\mathbb{B}}$, where B is a bisector with positive focus. In terms of [Gol, p. 248], the bisectors in our sense are extors, however, not every extor is a bisector in our sense.

We denote by $\mathrm{B}\wr g_1, g_2\wr$ the bisector with the real spine $\mathrm{G}\wr g_1, g_2\wr$. When using the notation $\mathrm{B}\wr g_1, g_2\wr$, we assume by default that $\operatorname{ta}(g_1, g_2) \neq 0, 1$.

polar point $\overset{\bullet}{f}$ to $\mathrm{L}\wr g_1, g_2\wr$
= focus of $\mathrm{B}\wr g_1, g_2\wr$

slice $\mathrm{L}\wr g, f\wr$

bisector $\mathrm{B}\wr g_1, g_2\wr$

g

g_1

complex spine $\mathrm{L}\wr g_1, g_2\wr$

real spine $\mathrm{G}\wr g_1, g_2\wr$ g_2

Lemma 4.1.13 implies the

Proposition 4.1.20. *The bisector* $\mathrm{B}\wr g_1, g_2\wr$ *is given in* $\mathbb{P}$ *by the equation* $b(x, g_1, g_2) = 0$. ∎

Lemma 4.1.21 [Gol, p. 165, Theorem 5.2.4]**.**
Let $p_1, p_2 \in \mathbb{P}$ *be such that* $\mathrm{ta}(p_1, p_2) \neq 0, 1$ *and
let* $S_i := \mathbb{P}p_i^\perp$, $i = 1, 2$. *Then there exists a
unique bisector* B *such that* S_1 *and* S_2 *are slices
of* B. *The real spine of such* B *is* $\mathrm{G}\wr p_1, p_2\wr$.

Proof. The intersection f of the S_i's has to
be the focus of a bisector B in question. By Lem-
ma 4.1.7, $\mathrm{L}\wr p_1, p_2\wr$ is necessarily the complex spi-
ne of B. Since the polar point to every slice of B

belongs to the real spine of B, this real spine should contain the points p_1 and p_2. By Remark 4.1.11,
the real spine of B has to equal $\mathrm{G}\wr p_1, p_2\wr$. By Remark 4.1.10, the point $g_i \in \mathrm{L}\wr p_1, p_2\wr$ orthogonal to p_i
belongs to $\mathrm{G}\wr p_1, p_2\wr$, $i = 1, 2$. In other words, $S_i = \mathrm{L}\wr g_i, f\wr$ is a slice of $\mathrm{B}\wr p_1, p_2\wr$. ∎
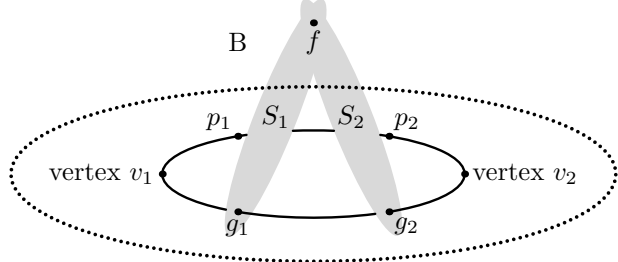
In particular, two ultraparallel complex geodesics S_1 and S_2 are slices of a unique bisector B. Obvi-
ously, the focus f of such B is positive. The projective cone over $\mathrm{G}[g_1, g_2]$ with vertex f is the segment
$\mathrm{B}[S_1, S_2]$ of B, where g_i stands for the intersection of S_i with the real spine of B. Using the parameter-
ization given in Lemma 4.1.15 of (positive) polar points to the slices of B, it is easy to show that there
exists a complex geodesic M called the *middle slice* of the segment $\mathrm{B}[S_1, S_2]$ such that the reflection in
M exchanges S_1 and S_2.

Lemma 4.1.22. *Suppose that the Gram matrix of* $p_1, p_2 \in V$ *has the form* $\left(\begin{smallmatrix} 1 & t \\ t & 1 \end{smallmatrix} \right)$ *with* $t > 1$. *Then*
$m := \dfrac{p_1 + p_2}{\sqrt{2t + 2}}$ *is the polar point to the middle slice* M *of the segment* $\mathrm{B}[S_1, S_2]$ *and* $\langle m, m \rangle = 1$, *where*
$S_i := \mathbb{P}p_i^\perp$, $i = 1, 2$.

Proof. It is easy to see that $\langle m, m \rangle = 1$. We have (see (2.1.3) for the definition of $R(m)$)

$$R(m)p_1 = 2\langle p_1, m \rangle m - p_1 = \frac{2\langle p_1, p_1 + p_2 \rangle}{2t + 2}(p_1 + p_2) - p_1 = p_2. \qquad \blacksquare$$

For different points $v_1, v_2 \in \partial_\infty \mathbb{B}$ and $p \notin \partial_\infty \mathbb{B}$, the η-invariant [Gol, p. 231] is defined as

$$\eta(v_1, v_2, p) := \frac{\langle v_1, p \rangle \langle p, v_2 \rangle}{\langle v_1, v_2 \rangle \langle p, p \rangle}. \tag{4.1.23}$$

The tance between a point and a bisector can be expressed in terms of the η-invariant:

Lemma 4.1.24 [San, p. 97, Example 2]**.** *Let* $\mathrm{B}\wr v_1, v_2\wr$ *be a bisector with vertices* $v_1, v_2 \in \partial_\infty \mathbb{B}$ *and
let* $p \in \mathbb{B} \setminus \mathrm{B}\wr v_1, v_2\wr$. *Then* $\mathrm{ta}\left(p, \mathrm{B}\wr v_1, v_2\wr \cap \mathbb{B} \right) = 1 - \mathrm{Re}\,\eta(v_1, v_2, p) + \left| \eta(v_1, v_2, p) \right|$.

Proof. We can assume that $\langle p, p \rangle = -1$, $\langle v_1, v_2 \rangle = \frac{1}{2}$, and $r := |z_1| = |z_2|$, where $z_i := \langle p, v_i \rangle$.
By Lemma 4.1.15, the polar points to the slices of signature $+-$ are parameterized by $g(t) := t^{-1}v_1 + tv_2$,
$t > 0$, with $\langle g(t), g(t) \rangle = 1$. By Lemma 4.1.8,

$$\mathrm{ta}\left(p, \mathbb{P}g(t)^\perp \cap \mathbb{B} \right) = 1 - \mathrm{ta}\left(p, g(t) \right) = 1 + 2\,\mathrm{Re}(\overline{z}_1 z_2) + t^{-2}r^2 + t^2 r^2.$$

This function has a unique minimum when $t = 1$. It remains to observe that $\eta(v_1, v_2, p) = -2\overline{z}_1 z_2$. ∎

4.2. Bisectors

In this subsection, we explicitly describe the tangent space (at generic points) to subspaces of the form $\mathbb{P}W$, where $W \subset V$ is an $\mathbb{R}$-vector subspace. This tool provides a handy description of the normal vector to a bisector. Then we prove some known facts concerning transversalities. Finally, we observe that a meridional curve (see 2.1.1) depends continuously on its segment of bisector and initial point.

4.2.1. Let $W \subset V$ be an $\mathbb{R}$-vector subspace. We define

$$d := \min_{0 \neq w \in W} \dim_{\mathbb{R}} \mathbb{C}w \cap W, \qquad D := \{w \in W \mid \dim_{\mathbb{R}} \mathbb{C}w \cap W = d\}.$$

It is easy to show that D is open in W, that $\mathbb{P}D$ is smooth and open in $\mathbb{P}W$, and that $\dim_{\mathbb{R}} \mathbb{P}D = \dim_{\mathbb{R}} W - d$.

Lemma 4.2.2. *Let $p \in D$ and let $\varphi \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}p, V)$. Then $t_\varphi \in \mathrm{T}_p \mathbb{P}W$ if and only if $\varphi(p) \in \mathbb{C}p + W$.*

Proof. Suppose that $\varphi(p) \in W$. For small $\varepsilon \in \mathbb{R}$, we have a smooth curve $c(\varepsilon) \in \mathbb{P}D$ whose lift to V is given by $c_0(\varepsilon) := p + \varepsilon\varphi(p) \in W$. Therefore, the tangent vector t_φ to c belongs to $\mathrm{T}_p \mathbb{P}D = \mathrm{T}_p \mathbb{P}W$. Such tangent vectors form the whole $\mathrm{T}_p \mathbb{P}D$ since

$$\dim_{\mathbb{R}} \mathbb{P}D = \dim_{\mathbb{R}} W - d = \dim_{\mathbb{R}} W/\mathbb{C}p \cap W. \qquad \blacksquare$$

Corollary 4.2.3. *Let L be a projective line, let $p \in \mathrm{L}$ be nonisotropic, and let $q \in \mathrm{L}$ be orthogonal to p. Then $\mathrm{T}_p \mathrm{L} = \langle -, p \rangle \mathbb{C}q$. Let $\mathbb{P}S$ be an $\mathbb{R}$-plane or geodesic and let $p \in S$ be nonisotropic. Then $\mathrm{T}_p \mathbb{P}S = \langle -, p \rangle (p^\perp \cap S)$.* $\blacksquare$

As it is easy to see, Lemma 2.1.13 follows from the second part of Corollary 4.2.3.

4.2.4. Orthogonal projective lines. Let L_1 be a projective line and let $p \in \mathrm{L}_1$ be nonisotropic. Using Corollary 4.2.3, it is easy to show that there exists a unique projective line L_2 passing through p and orthogonal to L_1 in the sense that $\mathrm{T}_p \mathrm{L}_1$ and $\mathrm{T}_p \mathrm{L}_2$ are orthogonal. Obviously, $\mathrm{L}_2 = \mathbb{P}p_1^\perp$ for some $p_1 \in \mathrm{L}_1$. This means that two distinct projective lines intersecting in a nonisotropic point p are orthogonal at p if and only if their polar points are orthogonal. In this case, $\mathrm{T}_p \mathbb{P} = \mathrm{T}_p \mathrm{L}_1 \oplus \mathrm{T}_p \mathrm{L}_2$. In particular, the complex spine of a bisector is orthogonal to every slice of signature different from $+0$. This implies that the intersection of such a slice with a meridian is a geodesic orthogonal to the real spine.

4.2.5. Some simple transversalities. In order to show a few simple facts concerning transversalities, we need the following

Lemma 4.2.6. *Let $p \in \partial_\infty \mathbb{B}$ and let $\varphi \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}p, V)$. Then $t_\varphi \in \mathrm{T}_p \partial_\infty \mathbb{B}$ if and only if $\mathrm{Re}\langle \varphi(p), p \rangle = 0$.*

Proof. We take $q \in V$ such that $\langle p, q \rangle \neq 0$. In some neighbourhood of p, the 3-sphere $\partial_\infty \mathbb{B}$ is given by the equation $h(x) := \dfrac{\langle x, x \rangle}{\langle q, x \rangle \langle x, q \rangle} = 0$. The tangent vector t_φ is simply $\dot{c}(0)$, where $c(\varepsilon)$ is a smooth curve in $\mathbb{P}$ whose lift to V is given by $c_0(\varepsilon) = p + \varepsilon\varphi(p)$. It follows from

$$\dot{c}(0)h = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} h\big(p + \varepsilon\varphi(p)\big) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\langle p + \varepsilon\varphi(p), p + \varepsilon\varphi(p) \rangle}{\langle q, p + \varepsilon\varphi(p) \rangle \langle p + \varepsilon\varphi(p), q \rangle} = \frac{\langle \varphi(p), p \rangle + \langle p, \varphi(p) \rangle}{\langle p, q \rangle \langle q, p \rangle}$$

that $\dot{c}(0)h = 0$ if and only if $\mathrm{Re}\langle \varphi(p), p \rangle = 0$. It remains to observe that such t_φ's form a 3-dimensional $\mathbb{R}$-vector subspace in $\mathrm{T}_p \mathbb{P}$. $\blacksquare$

Remark 4.2.7 (folklore). *Every projective line of signature $+-$ is transversal to $\partial_\infty \mathbb{B}$.*

Proof. Let $p \notin \overline{\mathbb{B}}$ and let $q \in \mathbb{P}p^\perp \cap \partial_\infty \mathbb{B}$. Fix some point $q' \in \mathbb{P}p^\perp$ different from q. So, $\mathbb{P}p^\perp = \mathrm{L} \, \wr q, q' \wr$ and we can assume that $0 \neq \langle q, q' \rangle \in \mathbb{R}$. Suppose that $t_\varphi \in \mathrm{T}_q \, \mathbb{P}p^\perp \cap \mathrm{T}_q \, \partial_\infty \mathbb{B}$, where $\varphi \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}q, V)$. By Lemma 4.2.2, $\varphi(q) \in p^\perp = \mathbb{C}q + \mathbb{C}q'$. We can change φ by adding to $\varphi(q)$ an arbitrary element in $\mathbb{C}q$ since this does not alter t_φ. Hence, we can assume that $\varphi(q) \in \mathbb{C}q'$. By Lemma 4.2.6, $\varphi(q) \in \mathbb{R}iq'$. $\blacksquare$

Corollary 4.2.8. *Every bisector* B *is transversal to* $\partial_\infty \mathbb{B}$.

Proof. Denote by $\mathrm{G} \, S$ the real spine of B and by f, the focus of B. Let $p \in \mathrm{B} \cap \partial_\infty \mathbb{B}$. Obviously, $p \neq f$. We can assume $p = s + cf$ for some $0 \neq s \in S$ and $c \in \mathbb{C}$. It follows from $p \in \partial_\infty \mathbb{B}$ that $\langle s, s \rangle + |c|^2 \langle f, f \rangle = 0$. Take some $s' \in S \setminus \mathbb{R}s$.

Suppose that $t_\varphi \in \mathrm{T}_p \, \mathrm{B} \cap \mathrm{T}_p \, \partial_\infty \mathbb{B}$, where $\varphi \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}p, V)$. By Lemmas 4.2.2 and 4.2.6, $\varphi(p) \in \mathbb{C}s + \mathbb{R}s' + \mathbb{C}f$ and $\mathrm{Re} \, \langle \varphi(p), p \rangle = 0$. Adding to $\varphi(p)$ an arbitrary element from $\mathbb{C}p$ does not change t_φ. Therefore, we can assume that $\varphi(p) \in \mathbb{R}s' + \mathbb{C}f$.

If $c = 0$, then $p = s$ is isotropic and $\langle s', s \rangle \neq 0$. In this case, the equality $\mathrm{Re} \, \langle \varphi(p), p \rangle = 0$ implies $\varphi(p) \in \mathbb{C}f$ and we conclude that $\dim_{\mathbb{R}}(\mathrm{T}_p \, \mathrm{B} \cap \mathrm{T}_p \, \partial_\infty \mathbb{B}) \leq 2$. If $c \neq 0$, then s is nonisotropic and we can assume that $c = 1$ and take $s' \in S \setminus \mathbb{R}s$ orthogonal to s. In this case, the equality $\mathrm{Re} \, \langle \varphi(p), p \rangle = 0$ implies $\varphi(p) \in \mathbb{R}s' + \mathbb{R}if$. $\blacksquare$

4.2.9. We now introduce the equation of the tangent space to the bisector $\mathrm{B} \, \wr g_1, g_2 \wr$. Define

$$t(v, p, g_1, g_2) := \frac{\langle g_1, v \rangle \langle p, g_2 \rangle + \langle g_1, p \rangle \langle v, g_2 \rangle}{\langle g_1, g_2 \rangle} - \frac{\langle g_2, v \rangle \langle p, g_1 \rangle + \langle g_2, p \rangle \langle v, g_1 \rangle}{\langle g_2, g_1 \rangle}.$$

Note that $t(v, p, g_1, g_2)$ does not depend on the choice of representatives $v, p, g_1, g_2 \in V$ that provides the same linear map $\langle -, p \rangle v$.

Lemma 4.2.10. *Let* $p \in \mathrm{B} \, \wr g_1, g_2 \wr$ *be different from the focus* f *of* $\mathrm{B} \, \wr g_1, g_2 \wr$ *and let* $\varphi \in \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}p, V)$. *Then* $t_\varphi \in \mathrm{T}_p \, \mathrm{B} \, \wr g_1, g_2 \wr$ *if and only if* $t\big(\varphi(p), p, g_1, g_2\big) = 0$.

Proof. The equality $t\big(\varphi(p), p, g_1, g_2\big) = 0$ does not change its sense if we choose other representatives $g_1, g_2, p \in V$. Hence, we can assume that $g_1, g_2 \in S$ and $p \in W$, where $\mathrm{G} \, \wr g_1, g_2 \wr = \mathrm{G} \, S$, $\mathrm{B} \, \wr g_1, g_2 \wr = \mathbb{P}W$, and $W := S + \mathbb{C}f$. By Lemma 4.2.2, $t_\varphi \in \mathrm{T}_p \, \mathrm{B} \, \wr g_1, g_2 \wr$ if and only if $\varphi(p) \in \mathbb{C}p + W$.

Suppose that $\varphi(p) \in \mathbb{C}p + W$. Let us show that $t\big(\varphi(p), p, g_1, g_2\big) = 0$. Clearly, $t(s, p, g_1, g_2) = 0$ for all $s \in S$. Since $\langle g_i, f \rangle = 0$ for $i = 1, 2$, we obtain $t(x, p, g_1, g_2) = 0$ for all $x \in \mathbb{C}f$. For every $c \in \mathbb{C}$,

$$t(cp, p, g_1, g_2) = \frac{\overline{c} \langle g_1, p \rangle \langle p, g_2 \rangle + c \langle g_1, p \rangle \langle p, g_2 \rangle}{\langle g_1, g_2 \rangle} - \frac{\overline{c} \langle g_2, p \rangle \langle p, g_1 \rangle + c \langle g_2, p \rangle \langle p, g_1 \rangle}{\langle g_2, g_1 \rangle} = (c + \overline{c}) b(p, g_1, g_2) = 0$$

by Proposition 4.1.20. Hence, $t\big(\varphi(p), p, g_1, g_2\big) = 0$.

Conversely, suppose that $t\big(\varphi(p), p, g_1, g_2\big) = 0$. The point p is different from f. Interchanging g_1 and g_2 if necessary, we can assume that $p = g_1 + rg_2 + c'f$ for some $r \in \mathbb{R}$ and $c' \in \mathbb{C}$. As was shown, $t(\mathbb{C}p + \mathbb{C}f, p, g_1, g_2) = 0$. It is easy to see that $V = \mathbb{C}g_2 + \mathbb{C}p + \mathbb{C}f$. Hence, we can assume that $\varphi(p) = cg_2$ with $c \in \mathbb{C}$. Since $t(cg_2, g_2, g_1, g_2) = (c + \overline{c}) b(g_2, g_1, g_2) = 0$ and $\langle g_1, g_2 \rangle \in \mathbb{R}$, we obtain

$$t(cg_2, p, g_1, g_2) = t(cg_2, g_1, g_1, g_2) = (\overline{c} - c) \frac{\langle g_1, g_2 \rangle \langle g_2, g_1 \rangle - \langle g_1, g_1 \rangle \langle g_2, g_2 \rangle}{\langle g_1, g_2 \rangle}.$$

By Sylvester's Criterion, $t(cg_2, p, g_1, g_2) = 0$ implies $c \in \mathbb{R}$. $\blacksquare$

Proposition 4.2.11. *Let* $p \in \mathrm{B} \, \wr g_1, g_2 \wr$ *be nonisotropic and different from the focus* f *of* $\mathrm{B} \, \wr g_1, g_2 \wr$ *and let* $g \in \mathrm{G} \, \wr g_1, g_2 \wr$ *stand for the polar point to the slice of* $\mathrm{B} \, \wr g_1, g_2 \wr$ *containing* p. *Then*

$$0 \neq n(p, g_1, g_2) := \langle -, p \rangle i \Big(\frac{\langle p, g_2 \rangle}{\langle g_1, g_2 \rangle} g_1 - \frac{\langle p, g_1 \rangle}{\langle g_2, g_1 \rangle} g_2 \Big) \in \langle -, p \rangle \mathbb{C}g$$

is a normal vector to $\mathrm{B}\wr g_1, g_2\wr$ at p.

Proof. If $n(p, g_1, g_2) = 0$, then $\langle p, g_i \rangle = 0$ for $i = 1, 2$, implying $p = f$. A contradiction.

Obviously, $n(p, g_1, g_2) \in \langle -, p \rangle f^\perp$. By Proposition 4.1.20, $b(p, g_1, g_2) = 0$, implying $n(p, g_1, g_2) \in \langle -, p \rangle p^\perp$. Clearly, $\mathrm{L}(p, f)$ is the slice of $\mathrm{B}\wr g_1, g_2\wr$ containing p. Therefore, $p^\perp \cap f^\perp = \mathbb{C}g$.

Take an arbitrary vector $v_p \in \mathrm{T}_p \mathbb{P}$, $v \in p^\perp$ (see (4.1.3) for the definition). By Lemma 4.2.10, $v_p \in \mathrm{T}_p \mathrm{B}\wr g_1, g_2\wr$ if and only if $t(v, p, g_1, g_2) = 0$. Since

$$\big\langle n(p, g_1, g_2), v_p \big\rangle = -\langle p, p \rangle i \Big(\frac{\langle g_1, v \rangle \langle p, g_2 \rangle}{\langle g_1, g_2 \rangle} - \frac{\langle g_2, v \rangle \langle p, g_1 \rangle}{\langle g_2, g_1 \rangle} \Big)$$

and p is nonisotropic, the equality $t(v, p, g_1, g_2) = 0$ is equivalent to $\mathrm{Re} \big\langle n(p, g_1, g_2), v_p \big\rangle = 0$. ∎

4.2.12. We introduce the orientation of $\mathrm{G}\wr g_1, g_2\wr$ as follows. Let $i = 1, 2$. If $g_i \in \mathbb{B}$, then define $q_i \in \mathrm{G}\wr g_1, g_2\wr$ as the point orthogonal to g_i. Otherwise, put $q_i := g_i$. Denote by $a \subset \mathrm{G}\wr g_1, g_2\wr$ the open arc from q_2 to q_1 that contains no point orthogonal to q_i, $i = 1, 2$, and such that $a \cap \overline{\mathbb{B}} = \varnothing$. The *orientation* of $\mathrm{G}\wr g_1, g_2\wr$ is that of a.

Remark 4.2.13. *Let $\mathrm{G}\wr g_1, g_2\wr$ be a geodesic, let $p_i \in \mathrm{G}\wr g_1, g_2\wr$ be orthogonal to g_i, and let p denote the polar point to $\mathrm{L}\wr g_1, g_2\wr$. Suppose that $p_1, p_2 \notin \mathbb{B}$. We have the following cases:*

- *$p \in \mathbb{B}$. Then g_1, g_2, p_1, p_2 are in a cyclic order in the circle $\mathrm{G}\wr g_1, g_2\wr$.*
- *$p \notin \overline{\mathbb{B}}$. Then $g_1, g_2, v_2, p_2, p_1, v_1$ are in a cyclic order in the circle $\mathrm{G}\wr g_1, g_2\wr$, where v_1, v_2 are the vertices of $\mathrm{G}\wr g_1, g_2\wr$ taken in an appropriate order. (Note that we admit $g_i = v_i$ or $p_i = v_i$.)* ∎

The bisector $\mathrm{B}\wr g_1, g_2\wr$ is *oriented* with respect to the orientation of its real spine $\mathrm{G}\wr g_1, g_2\wr$ and the natural (complex) orientation of its slices. Using Remark 4.2.13, it is easy to see that the introduced orientation of a bisector is compatible to the one introduced in 2.1.1.

4.2.14. We will show that the normal vector in Proposition 4.2.11 is in fact normal to the oriented bisector $\mathrm{B}\wr g_1, g_2\wr$. It follows from Proposition 4.1.20 that every bisector $\mathrm{B}\wr g_1, g_2\wr$ divides $\mathbb{P}$ into two *extended* half-spaces given by the inequalities

$$\mathrm{Im} \frac{\langle g_1, x \rangle \langle x, g_2 \rangle}{\langle g_1, g_2 \rangle} \geq 0, \qquad \mathrm{Im} \frac{\langle g_1, x \rangle \langle x, g_2 \rangle}{\langle g_1, g_2 \rangle} \leq 0.$$

Lemma 4.2.15. *Let $\mathrm{B}\wr g_1, g_2\wr$ be an oriented bisector, let $p_i \in \mathrm{G}\wr g_1, g_2\wr$ be the point orthogonal to g_i, $i = 1, 2$, and let $p \in \mathrm{B}\wr g_1, g_2\wr$ be a nonisotropic point different from the focus f of $\mathrm{B}\wr g_1, g_2\wr$. Denote by σ_f the signature of f. Then*

- *the inequalities $\mathrm{Im} \dfrac{\langle g_1, x \rangle \langle x, g_2 \rangle}{\langle g_1, g_2 \rangle} \geq 0$ and $\mathrm{Im} \dfrac{\langle p_1, x \rangle \langle x, p_2 \rangle}{\langle p_1, p_2 \rangle} \geq 0$ are equivalent,*
- *the vector $n(p, g_1, g_2)$ is normal to the oriented bisector $\mathrm{B}\wr g_1, g_2\wr$,*
- *the extended half-space given by $\sigma_f \mathrm{Im} \dfrac{\langle g_1, x \rangle \langle x, g_2 \rangle}{\langle g_1, g_2 \rangle} \geq 0$ lies on the side of $n(p, g_1, g_2)$.*

Proof. By Remark 4.2.13, $\mathrm{G}\wr g_1, g_2\wr$ and $\mathrm{G}\wr p_1, p_2\wr$ coincide as oriented geodesics. By Proposition 4.1.20, we can exclude the case $x \in \mathrm{B}\wr g_1, g_2\wr = \mathrm{B}\wr p_1, p_2\wr$ and reduce the first assertion to the last two.

Fix $x \notin \mathrm{B}\wr g_1, g_2\wr$. We will vary p, g_1, g_2 keeping the following conditions: p belongs to the bisector and satisfies $f \neq p \notin \partial_\infty \mathbb{B}$; g_1, g_2 belong to the real spine and satisfy $\mathrm{ta}(g_1, g_2) \neq 0, 1$. During the deformation, the orientation of $\mathrm{B}\wr g_1, g_2\wr$ remains the same, the normal vector $n(p, g_1, g_2)$ never vanishes, and $\mathrm{Im} \dfrac{\langle g_1, x \rangle \langle x, g_2 \rangle}{\langle g_1, g_2 \rangle} \neq 0$. Therefore, we can assume that $p \in \mathrm{G}\wr g_1, g_2\wr$, that g_1, g_2 are vertices if $\sigma_f = +1$, and that $p = g_2$ if $\sigma_f = -1$.

The case $\sigma_f = +1$. By Lemma 4.1.15, we can choose representatives so that $\langle g_1, g_2 \rangle = \pm\frac{1}{2}$ and $p = g_1 + g_2$. By Proposition 4.2.11, $n(p, g_1, g_2) = \langle -, p \rangle i (g_1 - g_2)$. By Lemma 4.1.15, a lift of the oriented geodesic G $\wr g_1, g_2 \wr$ is given by $c_0(t) := e^{\pm t} g_1 + e^{\mp t} g_2$, $t \in \mathbb{R}$. Obviously, $c_0(0) = p$. By Lemma 4.1.4, $\dot{c}(0) = \langle -, p \rangle (g_1 - g_2)$. Since $i\dot{c}(0) = n(p, g_1, g_2)$, the vector $n(p, g_1, g_2)$ is normal to the oriented bisector B $\wr g_1, g_2 \wr$.

The curve $n_0(t) := p + ti\langle p, p \rangle (g_1 - g_2) = (1 \pm ti) g_1 + (1 \mp ti) g_2$ is a lift of a smooth curve $n(t) \in \mathbb{P}$ with $\dot{n}(0) = n(p, g_1, g_2)$. It remains to observe that Im $\dfrac{\langle g_1, n(t) \rangle \langle n(t), g_2 \rangle}{\langle g_1, g_2 \rangle} = t$.

The case $\sigma_f = -1$. We can choose representatives so that $\langle g_i, g_i \rangle = 1$ and $\langle g_1, g_2 \rangle = r > 0$. By Sylvester's Criterion, $r < 1$. By Proposition 4.2.11, $n(p, g_1, g_2) = \langle -, p \rangle i (\frac{1}{r} g_1 - g_2)$. A lift of the oriented geodesic G $\wr g_1, g_2 \wr$ is given by $c_0(t) := (1 - t) g_2 + t g_1$, $t \in [0, 1]$. (Note that $\langle c_0(t), g_2 \rangle = 0$ would imply $t = \frac{1}{1-r} > 1$.) By Lemma 4.1.4, $\dot{c}(0) = \langle -, p \rangle (g_1 - r g_2)$. The vector $n(p, g_1, g_2)$ is normal to the oriented bisector B $\wr g_1, g_2 \wr$ because $i\dot{c}(0) = r n(p, g_1, g_2)$.

The curve $n_0(t) := g_2 + ti(g_1 - r g_2) = ti g_1 + (1 - rti) g_2$ is a lift of a smooth curve $n(t) \in \mathbb{P}$ with $\dot{n}(0) = r n(p, g_1, g_2)$. It remains to observe that Im $\dfrac{\langle g_1, n(t) \rangle \langle n(t), g_2 \rangle}{\langle g_1, g_2 \rangle} = -t\left(\dfrac{1 - r^2}{r}\right)$. ∎

Lemma 4.2.16. *An* (ideal) *meridional curve depends continuously on its segment of bisector and initial point.*

Proof. Let B$[g_1, g_2]$ be a segment of bisector with focus $f \notin \overline{\mathbb{B}}$ and $g_1, g_2 \in \mathbb{B}$, let S_1 denote the initial slice of B$[g_1, g_2]$, i.e., the one containing g_1, let $p_1 \in$ G $\wr g_1, g_2 \wr$ be orthogonal to g_1, and let $m_0 \in$ G $\wr g_1, g_2 \wr$ denote the polar point to the middle slice of B$[g_1, g_2]$. The meridional curve b of B$[g_1, g_2]$ generated by $p \in S_1 \cap \overline{\mathbb{B}}$ can be parameterized by G$[g_1, g_2]$ as well as by G$[p_1, m_0]$, where $m \in$ G$[p_1, m_0]$ is the polar point to the middle slice of B$[g_1, g]$ (in other words, the parameters g, m are related by the equality $g = R(m) g_1$). So, by Lemma 2.1.4,

$$b(g) := -R(m)p.$$

Since $p \neq f$, we can take p in the form $p = cf + g_1$ with $c \in \mathbb{C}$. We assume that $\langle g_1, g_1 \rangle = -1$. Then $\langle g, g \rangle = -1$ because $g = R(m) g_1$. It follows from $\langle m, m \rangle > 0$ and (2.1.3) that

$$\langle g, g_1 \rangle = \langle R(m) g_1, g_1 \rangle = 2\frac{\langle g_1, m \rangle \langle m, g_1 \rangle}{\langle m, m \rangle} - \langle g_1, g_1 \rangle = 2\frac{\langle g_1, m \rangle \langle m, g_1 \rangle}{\langle m, m \rangle} + 1 > 0,$$

implying $\sqrt{\mathrm{ta}(g, g_1)}\dfrac{\langle p, g_1 \rangle}{\langle g, g_1 \rangle} = -1$. We have $R(m)f = -f$ since $\langle m, f \rangle = 0$. It follows that $b(g) = -R(m)p = cf - g = \pi[g_1]p - g$. Therefore,

$$b(g) = \pi[g_1]p + \sqrt{\mathrm{ta}(g, g_1)}\frac{\langle p, g_1 \rangle}{\langle g, g_1 \rangle}g.$$

Note that this formula does not depend on the choice of representatives $g_1, g \in V$ and is linear in p. ∎

4.3. Bisectors with common slice

Using the formula of the normal vector to a bisector, we calculate in this subsection the angle between two cotranchal (= having a common slice) bisectors at a point in their common slice (for a geometric interpretation of this angle, see [AGr1]). Then we present a numerical criterion of transversality of such pair of bisectors. Finally, we prove some properties of transversal cotranchal bisectors including Condition (3) of [AGr3, Theorem 3.5].

Lemma 4.3.1. *Let $g \in \mathbb{P} \setminus \partial_\infty \mathbb{B}$ and $g_1, g_2 \in \mathbb{P}$ be such that $\mathrm{ta}(g, g_i) \neq 0, 1$ and let $p \in \mathbb{P}g^\perp$ be nonisotropic and different from the foci of $\mathrm{B}\langle g, g_i \rangle$, $i = 1, 2$. Then*

$$\langle n(p, g, g_2), n(p, g, g_1) \rangle = -\langle p, p \rangle \langle g, g \rangle \frac{\langle g_1, p \rangle \langle p, g_2 \rangle}{\langle g_1, g \rangle \langle g, g_2 \rangle}.$$

Proof. By Proposition 4.2.11, $n(p, g, g_1) = \langle -, p \rangle i \dfrac{\langle p, g_1 \rangle}{\langle g, g_1 \rangle} g$ and $n(p, g, g_2) = \langle -, p \rangle i \dfrac{\langle p, g_2 \rangle}{\langle g, g_2 \rangle} g$. The result follows from (4.1.2). ∎

The normal vectors $n(p, g, g_1)$ and $n(p, g, g_2)$ are both tangent to the naturally oriented projective line that passes through p and is orthogonal to the common slice $\mathbb{P}g^\perp$ of the oriented bisectors $\mathrm{B}\langle g, g_i \rangle$, $i = 1, 2$. Hence, the oriented angle from $n(p, g, g_1)$ to $n(p, g, g_2)$ is simply $\mathrm{Arg}\langle n(p, g, g_2), n(p, g, g_1) \rangle$. (Recall that the function Arg takes values in $[0, 2\pi)$.)

Corollary 4.3.2. *Let $\mathcal{E} \in \mathrm{SU}$ be a regular elliptic isometry and let $e_1, e_2, e_3 \in \mathbb{P}$ be the points corresponding to the eigenvectors of $\mathcal{E}$ such that $e_1 \in \mathbb{B}$. Denote by E the complex geodesic with the polar point e_2. Let D be a complex geodesic ultraparallel to E. Then the oriented angle from $\mathrm{B}[E, D]$ to $\mathrm{B}[E, \mathcal{E}D]$ at $e_1 \in E$ equals $\mathrm{Arg}(\xi_2 \xi_1^{-1})$, where ξ_i stands for the eigenvalue of $\mathcal{E}$ corresponding to e_i, $i = 1, 2$.*

Proof. Denote by d the polar point to D. The points d and $e_1 \in E$ cannot be orthogonal, nor can d and e_2, since D and E are ultraparallel. Therefore, we can choose representatives of $e_1, e_2 \in V$ so that $\langle d, e_1 \rangle > 0$ and $\langle d, e_2 \rangle > 0$. By Lemma 4.3.1, the angle in question equals

$$\mathrm{Arg}\, \frac{\langle d, e_1 \rangle \langle e_1, \mathcal{E}d \rangle}{\langle d, e_2 \rangle \langle e_2, \mathcal{E}d \rangle} = \mathrm{Arg}\, \frac{\langle \mathcal{E}^{-1}e_1, d \rangle}{\langle \mathcal{E}^{-1}e_2, d \rangle} = \mathrm{Arg}\, \frac{\xi_1^{-1} \langle e_1, d \rangle}{\xi_2^{-1} \langle e_2, d \rangle} = \mathrm{Arg}(\xi_2 \xi_1^{-1}). \qquad \blacksquare$$

Criterion 4.3.3. *Let $g \notin \overline{\mathbb{B}}$ and $g_1, g_2 \in \mathbb{P}$ be such that $\mathrm{ta}(g, g_i) > 1$ for $i = 1, 2$. The bisectors $\mathrm{B}\langle g, g_1 \rangle$ and $\mathrm{B}\langle g, g_2 \rangle$ are transversal along $\mathbb{P}g^\perp \cap \overline{\mathbb{B}}$ if and only if*

$$\left| \mathrm{Re}\, \frac{\langle g_1, g_2 \rangle \langle g, g \rangle}{\langle g_1, g \rangle \langle g, g_2 \rangle} - 1 \right| < \sqrt{1 - \frac{1}{\mathrm{ta}(g, g_1)}} \cdot \sqrt{1 - \frac{1}{\mathrm{ta}(g, g_2)}}.$$

Proof. We can choose representatives such that $\langle g, g \rangle = \langle g_1, g \rangle = \langle g, g_2 \rangle = 1$. Hence, $g_i = g + b_i$ with $b_i \in g^\perp$. (Since $b_i \in \mathrm{G}\langle g, g_i \rangle \cap \mathbb{P}g^\perp$, the slice of $\mathrm{B}\langle g, g_i \rangle$ containing b_i is $\mathbb{P}g^\perp$.) The inequality $\mathrm{ta}(g, g_i) > 1$ implies $\langle b_i, b_i \rangle < 0$. Normalizing the b_i's, we obtain $d_i := \dfrac{b_i}{\sqrt{-\langle b_i, b_i \rangle}}$ with $\langle d_i, d_i \rangle = -1$.

Let $d \in g^\perp$ be orthogonal to d_2, normalized so that $\langle d, d \rangle = 1$. Since d, d_2 form an orthonormal basis in $g^\perp$, we have $d_1 = rd + cd_2$ for suitable $r, c \in \mathbb{C}$. Choosing an appropriate representative $d \in V$, we can assume that $r \geq 0$. From $\langle d_1, d_1 \rangle = -1$, we obtain $r = \sqrt{|c|^2 - 1}$,

$$d_1 = \sqrt{|c|^2 - 1}\, d + cd_2, \qquad |c| \geq 1.$$

Every point $p \in \mathbb{P}g^\perp \cap \overline{\mathbb{B}}$ has the form $p = zd + d_2$ with $|z| \leq 1$. It follows from $\langle p, g_2 \rangle = \langle zd + d_2, g + b_2 \rangle = \langle d_2, b_2 \rangle$ that $0 \neq \langle p, g_2 \rangle \in \mathbb{R}$. Note that

$$\frac{\langle p, g_1 \rangle}{\sqrt{-\langle b_1, b_1 \rangle}} = \frac{\langle p, b_1 \rangle}{\sqrt{-\langle b_1, b_1 \rangle}} = \langle p, d_1 \rangle = \langle zd + d_2, \sqrt{|c|^2 - 1}\, d + cd_2 \rangle = z\sqrt{|c|^2 - 1} - \overline{c}.$$

Let $\varphi \in \mathrm{Lin}_\mathbb{C}(\mathbb{C}p, V)$. By Lemma 4.2.10, $t_\varphi \in \mathrm{T}_p \, \mathrm{B} \, \wr g, g_i \wr$ if and only if $t\big(\varphi(p), p, g, g_i\big) = 0$. In view of $p \in g^\perp$, this is equivalent to

$$\langle g, \varphi(p) \rangle \langle p, g_i \rangle \in \mathbb{R}.$$

By Lemma 4.2.2, $t_\varphi \in \mathrm{T}_p \, \mathbb{P}g^\perp$ if and only if $\varphi(p) \in g^\perp$. Therefore, considering t_φ modulo $\mathrm{T}_p \, \mathbb{P}g^\perp$, we can assume that $\varphi(p) \in \mathbb{C}g$.

Suppose that $\varphi(p) = ag \neq 0$, $a \in \mathbb{C}$. Since $0 \neq \langle p, g_2 \rangle \in \mathbb{R}$, the fact that $\langle g, \varphi(p) \rangle \langle p, g_2 \rangle \in \mathbb{R}$ is equivalent to $a \in \mathbb{R}$. Consequently, the condition $t_\varphi \in \mathrm{T}_p \, \mathrm{B} \, \wr g, g_1 \wr \cap \mathrm{T}_p \, \mathrm{B} \, \wr g, g_2 \wr$ is equivalent to the requirements that $a \in \mathbb{R}$ and $\langle p, g_1 \rangle \in \mathbb{R}$. In terms of z, the latter takes the form $z\sqrt{|c|^2 - 1} - \overline{c} \in \mathbb{R}$. The existence of z with $|z| \leq 1$ meeting the last condition means that $\sqrt{|c|^2 - 1} \geq |\mathrm{Im}\, c|$. This is equivalent to $|\mathrm{Re}\, c| \geq 1$. We can see that the bisectors $\mathrm{B} \wr g, g_1 \wr$ and $\mathrm{B} \wr g, g_2 \wr$ are transversal along $\mathbb{P}g^\perp \cap \overline{\mathbb{B}}$ if and only if $|\mathrm{Re}\, c| < 1$.

It remains to observe that $\sqrt{1 - \dfrac{1}{\mathrm{ta}(g, g_i)}} = \sqrt{1 - \langle g_i, g_i \rangle} = \sqrt{-\langle b_i, b_i \rangle}$ and that

$$-c\sqrt{-\langle b_1, b_1 \rangle}\sqrt{-\langle b_2, b_2 \rangle} = \langle d_1, d_2 \rangle \sqrt{-\langle b_1, b_1 \rangle}\sqrt{-\langle b_2, b_2 \rangle} = \langle b_1, b_2 \rangle = \langle g_1, g_2 \rangle - 1 = \frac{\langle g_1, g_2 \rangle \langle g, g \rangle}{\langle g_1, g \rangle \langle g, g_2 \rangle} - 1. \quad \blacksquare$$

In order to prove some fundamental properties of transversal cotranchal bisectors, we need the following technical

Remark 4.3.4. *Let $v_1, v_1', v_2, v_2' \in \partial_\infty \mathbb{B}$ be such that $v_i \neq v_i'$ for all $i = 1, 2$ and let $g \in \big(\mathrm{G} \wr v_1, v_1' \wr \cap \mathrm{G} \wr v_2, v_2' \wr \big) \setminus \overline{\mathbb{B}}$. Then there exist representatives $v_1, v_1', v_2, v_2', g \in V$ such that $\langle v_i, v_i' \rangle = \frac{1}{2}$ and $g = v_i + v_i'$ for all i. Such representatives satisfy the condition $\mathrm{Re}\langle v_1, v_2 \rangle \leq \frac{1}{4}$ or $\mathrm{Re}\langle v_1', v_2 \rangle \leq \frac{1}{4}$. Moreover, the inequality $\mathrm{Re}\langle v_1, v_2 \rangle \leq 2\big|\langle v_1, v_2 \rangle\big|^2$ holds, being the equality valid exactly when v_1, v_1', v_2, v_2', g belong to a same projective line.*

Proof. By Lemma 4.1.15, we take representatives $g_1, g_2 \in V$ of g and representatives $v_i, v_i' \in V$ such that $\langle v_i, v_i' \rangle = \frac{1}{2}$ and $g_i = v_i + v_i'$ for $i = 1, 2$. Since $|g_1| = |g_2|$, we have $g_1 = cg_2$ with $|c| = 1$. It remains to take cv_2, cv_2' in place of v_2, v_2'.

It follows from $\langle g, v_2 \rangle = \frac{1}{2}$ that $\langle v_1, v_2 \rangle + \langle v_1', v_2 \rangle = \frac{1}{2}$, implying that $\mathrm{Re}\langle v_1, v_2 \rangle \leq \frac{1}{4}$ or $\mathrm{Re}\langle v_1', v_2 \rangle \leq \frac{1}{4}$.

The rest follows from Sylvester's Criterion: the determinant of the Gram matrix of g, v_1, v_2 equals $\frac{1}{2}\mathrm{Re}\langle v_1, v_2 \rangle - \big|\langle v_1, v_2 \rangle\big|^2$. $\blacksquare$

Lemma 4.3.5 (compare with [Hsi, p. 97, Theorem 3.3]). *Let B_1 and B_2 be two bisectors with positive foci and let S be a common slice of B_1 and B_2 of signature $+-$. Suppose that B_1 and B_2 are transversal along $S \cap \overline{\mathbb{B}}$. Then $\mathrm{B}_1 \cap \overline{\mathbb{B}}$, $\mathrm{B}_2 \cap \overline{\mathbb{B}}$, and $\partial_\infty \mathbb{B}$ are transversal and $\mathrm{B}_1 \cap \mathrm{B}_2 \cap \overline{\mathbb{B}} = S \cap \overline{\mathbb{B}}$.*

Proof. Let $g \notin \overline{\mathbb{B}}$ denote the polar point to S and let $v_i, v_i' \in \partial_\infty \mathbb{B}$ denote the vertices of B_i, $i = 1, 2$. By Remark 4.3.4, we can assume that $\mathrm{Re}\, u \leq 2|u|^2$, $\mathrm{Re}\, u \leq \frac{1}{4}$, $\langle v_i, v_i' \rangle = \frac{1}{2}$, and $g = v_i + v_i'$ for $i = 1, 2$, where $u := \langle v_1, v_2 \rangle$. Since $\mathrm{B}_i = \mathrm{B} \wr g, v_i \wr$, we have $\mathrm{Re}\, u > 0$ by Criterion 4.3.3.

Every point in $\mathrm{G} \wr g, v_i \wr$ but v_i has the form

$$g_i(t_i) := g + (t_i - 1)v_i$$

with $t_i \in \mathbb{R}$. It is easy to see that $\langle g_i(t_i), g_i(t_i) \rangle = t_i$. Therefore, $g_i(t_i)$, $t_i > 0$, runs over all polar points to the slices of B_i of signature $+-$.

Let us show that the slices of signature $+-$ of the bisectors B_1 and B_2 that are different from their common slice S do not intersect in $\overline{\mathbb{B}}$. By Lemma 4.1.7, it suffices to prove that the inequality $\mathrm{ta}\big(g_1(t_1), g_2(t_2)\big) > 1$ is valid for $t_1, t_2 > 0$ unless $t_1 = t_2 = 1$. Since

$$2\big\langle g_1(t_1), g_2(t_2) \big\rangle = t_1 + t_2 + 2(t_1 - 1)(t_2 - 1)u$$

and $\langle g_i(t_i), g_i(t_i) \rangle = t_i$, we obtain

$$4t_1 t_2 \, \mathrm{ta}\big(g_1(t_1), g_2(t_2)\big) = (t_1 + t_2)^2 + 4(t_1 + t_2)(t_1 - 1)(t_2 - 1)\,\mathrm{Re}\,u + 4(t_1 - 1)^2(t_2 - 1)^2|u|^2.$$

Hence, the inequality $\mathrm{ta}\big(g_1(t_1), g_2(t_2)\big) > 1$ is equivalent to the inequality

$$(t_1 - t_2)^2 + 4(t_1 + t_2)(t_1 - 1)(t_2 - 1)\,\mathrm{Re}\,u + 4(t_1 - 1)^2(t_2 - 1)^2|u|^2 > 0$$

which can be rewritten in the form

$$(t_1 - t_2)^2 + 2(t_1^2 - 1)(t_2^2 - 1)\,\mathrm{Re}\,u + 2(t_1 - 1)^2(t_2 - 1)^2\big(2|u|^2 - \mathrm{Re}\,u\big) > 0.$$

Since $\mathrm{Re}\,u > 0$ and $2|u|^2 \geq \mathrm{Re}\,u$, this inequality is valid if $t_1, t_2 > 1$ or if $0 < t_1, t_2 < 1$. Therefore, we can assume that $0 < t_1 < 1 < t_2$. In this case, taking into account the inequalities $\mathrm{Re}\,u \leq \frac{1}{4}$ and $2|u|^2 \geq \mathrm{Re}\,u$, it suffices to observe that $(t_1 - t_2)^2 > \frac{1}{2}(1 - t_1^2)(t_2^2 - 1)$. We have shown that the slices of signature $+-$ of B_1 and B_2 do not intersect.

The bisectors B_1 and B_2 cannot have a common vertex v. Otherwise, they would have the same real spine $\mathrm{G}\wr g, v\wr$ and would coincide.

Note that u cannot be real. This would contradict the inequalities $0 < \mathrm{Re}\,u \leq \frac{1}{4}$ and $\mathrm{Re}\,u \leq 2|u|^2$. It is easy to see that no vertex of one bisector belongs to the other bisector. Indeed, $v_2 \in \mathrm{B}_1$ means that $\langle g_1(t_1), v_2 \rangle = 0$ for a suitable $t_1 > 0$. This can be written as $\frac{1}{2} + (t_1 - 1)u = 0$, implying that u is real. Similarly, $v_2' \in \mathrm{B}_1$ implies that $\frac{t_1}{2} + (1 - t_1)u = 0$ for a suitable $t_1 > 0$.

Thus, $\mathrm{B}_1 \cap \mathrm{B}_2 \cap \overline{\mathbb{B}} = S \cap \overline{\mathbb{B}}$.

The transversality follows from Corollary 4.2.8 and Remark 4.2.7.   ∎

Lemma 4.3.6. *Let B_1 and B_2 be bisectors with positive foci. Suppose that they possess a common slice S of signature $+-$ and that they are transversal along $S \cap \overline{\mathbb{B}}$. Then, for any $\vartheta > 0$, there exists some $\varepsilon > 0$ such that, for every $p \in \mathrm{B}_2 \cap \mathbb{B}$, the inequality $\mathrm{ta}(p, \mathrm{B}_1 \cap \mathbb{B}) < 1 + \varepsilon^2$ implies the inequality $\mathrm{ta}(p, S \cap \mathbb{B}) < 1 + \vartheta^2$.*

Proof. Let $g \notin \overline{\mathbb{B}}$ denote the polar point to S and let $v_i, v_i' \in \partial_\infty \mathbb{B}$ denote the vertices of B_i, $i = 1, 2$. By Remark 4.3.4, we can assume that $\mathrm{Re}\,u \leq 2|u|^2$, $\mathrm{Re}\,u \leq \frac{1}{4}$, $\langle v_i, v_i' \rangle = \frac{1}{2}$, and $g = v_i + v_i'$ for $i = 1, 2$, where $u := \langle v_1, v_2 \rangle$. In particular, we have $\langle v_i, g \rangle = \langle v_i', g \rangle = \frac{1}{2}$. (The equality $\mathrm{Re}\,u = 2|u|^2$ means that B_1 and B_2 have the same complex spine. In this case, the fact is easier and will be proven later.) Since $\mathrm{B}_i = \mathrm{B}\wr g, v_i\wr$, we have $\mathrm{Re}\,u > 0$ by Criterion 4.3.3.

We assume that $\mathrm{Re}\,u < 2|u|^2$. Therefore, g, v_1, v_2 are $\mathbb{C}$-linearly independent. The point

$$f := v_1 + (4u - 1)v_2 - 2ug$$

is the focus of B_2 because $\langle f, v_2 \rangle = \langle f, g \rangle = 0$. Define

$$k := \sqrt{4|u|^2 - 2\,\mathrm{Re}\,u}.$$

It is easy to see that

$$\langle f, f \rangle = \langle v_1, f \rangle = (4\overline{u} - 1)u - \overline{u} = k^2, \qquad \langle f, v_1' \rangle = \langle f, g - v_1 \rangle = -k^2.$$

Let us parameterize the points in $\mathrm{B}_2 \cap \mathbb{B}$. By Lemma 4.1.15, the negative points in the real spine $\mathrm{G}\wr v_2, v_2'\wr$ of B_2 are parameterized by $g(t) := t^{-1}v_2 - tv_2'$, $t > 0$. Since $\left\langle \dfrac{f}{k}, \dfrac{f}{k} \right\rangle = 1$, every point in $\mathrm{B}_2 \cap \mathbb{B}$

takes the form $p(t, z) := \dfrac{z}{k} f + g(t)$, where $t > 0$ and $|z| < 1$. (The point $p(t, z)$ belongs to the slice of B_2 that contains $g(t)$.) Calculating straightforwardly, we obtain

$$\langle p(t, z), g \rangle = \langle g(t), g \rangle = \frac{t^{-1} - t}{2}, \qquad \langle v_1, g(t) \rangle = \langle v_1, t^{-1} v_2 - t(g - v_2) \rangle = (t^{-1} + t)u - \frac{t}{2},$$

$$\langle g(t), v_1' \rangle = \langle g(t), g - v_1 \rangle = \frac{t^{-1}}{2} - (t^{-1} + t)\overline{u}, \qquad \langle g(t), g(t) \rangle = -1, \qquad \langle p(t, z), p(t, z) \rangle = |z|^2 - 1,$$

$$\langle v_1, p(t, z) \rangle = k\overline{z} + (t^{-1} + t)u - \frac{t}{2}, \qquad \langle p(t, z), v_1' \rangle = -kz + \frac{t^{-1}}{2} - (t^{-1} + t)\overline{u}.$$

By Lemma 4.1.8,

$$\mathrm{ta}\,\big(p(t, z), S \cap \mathbb{B}\big) = 1 - \mathrm{ta}\,\big(p(t, z), g\big) = 1 + \frac{(t^{-1} - t)^2}{4\big(1 - |z|^2\big)} = 1 + h^2, \qquad h := \frac{|t^{-1} - t|}{2\sqrt{1 - |z|^2}}.$$

In order to express explicitly the tance $\mathrm{ta}\,\big(p(t, z), B_1 \cap \mathbb{B}\big)$, we need to calculate the η-invariant (4.1.23) :

$$\eta\big(v_1, v_1', p(t, z)\big) = \frac{\big(2(t^{-1} + t)u + 2k\overline{z} - t\big)\big(2(t^{-1} + t)\overline{u} + 2kz - t^{-1}\big)}{2\big(1 - |z|^2\big)} = a + ib, \qquad (\textbf{4.3.7})$$

where
$$a := \frac{1 + (t^{-1} + t)^2 k^2 + 4k^2 |z|^2 + 2(t^{-1} + t)k\big((4u_0 - 1)z_0 - 4u_1 z_1\big)}{2\big(1 - |z|^2\big)},$$

$$b := (t^{-1} - t) \cdot \frac{(t^{-1} + t)u_1 - k z_1}{1 - |z|^2}, \qquad u = u_0 - iu_1, \qquad z = z_0 - iz_1, \qquad u_0, u_1, z_0, z_1 \in \mathbb{R}.$$

We have to estimate the tance $\mathrm{ta}\,\big(p(t, z), B_1 \cap \mathbb{B}\big)$ in terms of h. To this aim, we introduce

$$c := |4u_0 - 1| + 4|u_1|, \qquad d := (4u_0 - 1)z_0 - 4u_1 z_1, \qquad e := \sqrt{1 - |z|^2},$$

rewrite a and b (using the equality $t^{-1} + t = 2\sqrt{e^2 h^2 + 1}$) as

$$a = \frac{1 + 8k^2 + 4k^2 e^2 h^2 - 4k^2 e^2 + 4kd\sqrt{e^2 h^2 + 1}}{2e^2}, \qquad b = \pm h \frac{4u_1 \sqrt{e^2 h^2 + 1} - 2k z_1}{e}, \qquad (\textbf{4.3.8})$$

and observe that the inequalities

$$0 < c, \qquad |d| \le c, \qquad 0 < e \le 1, \qquad 0 \le h, \qquad 0 < k < 2|u_1|$$

are valid. Only the last inequality requires a proof: It follows from $0 < u_0 \le \frac{1}{4} < \frac{1}{2}$ that $(\frac{1}{2} - 2u_0)^2 < \frac{1}{4}$. Hence, $k = \sqrt{4u_1^2 + 4u_0^2 - 2u_0} = \sqrt{4u_1^2 + (\frac{1}{2} - 2u_0)^2 - \frac{1}{4}} < 2|u_1|$.

Suppose that $\mathrm{ta}\,\big(p(t, z), B_1 \cap \mathbb{B}\big) < 1 + \varepsilon^2$ for some $\varepsilon > 0$. By Lemma 4.1.24, $\mathrm{ta}\,\big(p(t, z), B_1 \cap \mathbb{B}\big) = 1 - a + |a + ib|$ in view of (4.3.7). Therefore, $b^2 < \varepsilon^4 + 2a\varepsilon^2$, which can be written in the form

$$\big(4hu_1 \sqrt{e^2 h^2 + 1} - 2hk z_1\big)^2 - \big(2\varepsilon k \sqrt{e^2 h^2 + 1} + \varepsilon d\big)^2 < \varepsilon^2 \big(1 + 4k^2 + \varepsilon^2 e^2 - 4k^2 e^2 - d^2\big)$$

taking the expressions (4.3.8) into account. Using an inequality of the type $\big(|A|-|B|\big)^2 - \big(|C|+|D|\big)^2 \le (A-B)^2 - (C+D)^2$ and the fact that $|d| \le c$ and $e^2 \le 1$, we conclude that

$$\big(4h|u_1|\sqrt{e^2h^2+1} - 2hk|z_1|\big)^2 - \big(2\varepsilon k\sqrt{e^2h^2+1} + \varepsilon c\big)^2 < \varepsilon^2\big(1 + 4k^2 + \varepsilon^2\big).$$

In view of $|z_1| < 1 \le \sqrt{e^2h^2+1}$ and $k < 2|u_1|$, this implies that

$$\big(4h|u_1|\sqrt{e^2h^2+1} - 2hk\big)^2 - \big(2\varepsilon k\sqrt{e^2h^2+1} + \varepsilon c\big)^2 < \varepsilon^2\big(1 + 4k^2 + \varepsilon^2\big).$$

This inequality can be converted into

$$\Big(\big(4h|u_1| - 2\varepsilon k\big)\sqrt{e^2h^2+1} - (2hk + \varepsilon c)\Big)\Big(\big(4h|u_1| + 2\varepsilon k\big)\sqrt{e^2h^2+1} - (2hk - \varepsilon c)\Big) < \varepsilon^2\big(1 + 4k^2 + \varepsilon^2\big).$$
$$(\textbf{4.3.9})$$

Given $\vartheta > 0$, we have to find some $\varepsilon > 0$ such that the inequality $\mathrm{ta}(p, \mathrm{B}_1 \cap \mathbb{B}) < 1 + \varepsilon^2$ implies $h < \vartheta$. First, we require that $\varepsilon \le \vartheta$. We assume now that $4h|u_1| - 2\varepsilon k \ge 0$ because, otherwise, the inequality $h < \varepsilon$ follows from $0 < k < 2|u_1|$ and provides $h < \vartheta$. Next, we require that $\varepsilon \le \dfrac{4|u_1| - 2k}{c + 2k}\vartheta$. Now, we assume that $4h|u_1| - 2\varepsilon k \ge 2hk + \varepsilon c$ (otherwise, the inequality $h < \vartheta$ follows) and conclude that

$$\big(4h|u_1| - 2\varepsilon k\big)\sqrt{e^2h^2+1} - (2hk + \varepsilon c) \ge 0$$

in view of $\sqrt{e^2h^2+1} \ge 1$. Requiring that $\varepsilon \le 1$, we can deduce from (4.3.9) that

$$\Big(\big(4h|u_1| - 2\varepsilon k\big)\sqrt{e^2h^2+1} - (2hk + \varepsilon c)\Big)^2 < \varepsilon^2(2 + 4k^2),$$

which, in its turn, implies $4h|u_1| - 2\varepsilon k - 2hk - \varepsilon c < \varepsilon\sqrt{2 + 4k^2}$ due to $1 \le \sqrt{e^2h^2+1}$ and $4h|u_1| - 2\varepsilon k \ge 2hk + \varepsilon c$. We obtain $h\big(4|u_1| - 2k\big) < \varepsilon(\sqrt{2 + 4k^2} + 2k + c)$ and, therefore, $h < \dfrac{\sqrt{2 + 4k^2} + 2k + c}{4|u_1| - 2k}\varepsilon$. Finally, we require that $\varepsilon \le \dfrac{4|u_1| - 2k}{\sqrt{2 + 4k^2} + 2k + c}\vartheta$.

The complete list of requirements concerning ε is the following:

$$\varepsilon \le \vartheta, \qquad \varepsilon \le \frac{4|u_1| - 2k}{c + 2k}\vartheta, \qquad \varepsilon \le 1, \qquad \varepsilon \le \frac{4|u_1| - 2k}{\sqrt{2 + 4k^2} + 2k + c}\vartheta.$$

(Note that none of these inequalities involves t or z.)

Now, we consider the case when B_1 and B_2 have the same complex spine. Let f be the common focus of B_1 and B_2. We assume $\langle f, f \rangle = 1$. Every point in $\mathrm{B}_2 \cap \mathbb{B}$ has the form $p(t, z) := zf + t^{-1}v_2 - tv_2'$, $t > 0$, $|z| < 1$. We have

$$\langle p(t, z), g \rangle = \frac{t^{-1} - t}{2}, \qquad \langle v_1, p(t, z) \rangle = \langle v_1, t^{-1}v_2 - t(g - v_2) \rangle = (t^{-1} + t)u - \frac{t}{2},$$

$$\langle p(t, z), v_1' \rangle = \langle p(t, z), g - v_1 \rangle = \frac{t^{-1}}{2} - (t^{-1} + t)\overline{u}, \qquad \langle p(t, z), p(t, z) \rangle = |z|^2 - 1.$$

By Lemma 4.1.8,

$$\mathrm{ta}\big(p(t, z), S \cap \mathbb{B}\big) = 1 - \mathrm{ta}\big(p(t, z), g\big) = 1 + \frac{(t^{-1} - t)^2}{4(1 - |z|^2)} = 1 + h^2, \qquad h := \frac{|t^{-1} - t|}{2\sqrt{1 - |z|^2}}.$$

Taking into account that $\operatorname{Re} u = 2|u|^2$, we calculate the η-invariant (4.1.23) :

$$\eta\big(v_1, v_1', p(t, z)\big) = \frac{\big(2(t^{-1} + t)u - t\big)\big(2(t^{-1} + t)\overline{u} - t^{-1}\big)}{2\big(1 - |z|^2\big)} = \frac{1 + 2i(t^2 - t^{-2})u_1}{2\big(1 - |z|^2\big)} = a + ib, \qquad (\textbf{4.3.10})$$

where

$$a := \frac{1}{2\big(1 - |z|^2\big)}, \qquad b := \frac{(t^2 - t^{-2})u_1}{1 - |z|^2}, \qquad u_1 := \operatorname{Im} u.$$

Suppose that $\operatorname{ta}\big(p(t, z), B_1 \cap \mathbb{B}\big) < 1 + \varepsilon^2$ for some $\varepsilon > 0$. By Lemma 4.1.24, $\operatorname{ta}\big(p(t, z), B_1 \cap \mathbb{B}\big) = 1 - a + |a + ib|$ in view of (4.3.10). Therefore, $b^2 < \varepsilon^4 + 2a\varepsilon^2$, which can be written in the form

$$\frac{(t^{-2} - t^2)^2 u_1^2}{1 - |z|^2} < \varepsilon^2\Big(\varepsilon^2\big(1 - |z|^2\big) + 1\Big).$$

Since $|z| < 1$ and $t + t^{-1} \ge 2$, we obtain $16h^2 u_1^2 < \varepsilon^2(\varepsilon^2 + 1)$. Note that u cannot be real: this would contradict $\operatorname{Re} u = 2|u|^2$ and $0 < \operatorname{Re} u \le \frac{1}{4}$. Requiring that $\varepsilon^2 \le 15$ and $\varepsilon \le |u_1|\vartheta$, we arrive at $h < \vartheta$. $\blacksquare$

4.4. Triangles of bisectors

This subsection begins with a numerical criterion of transversality of an oriented triangle of bisectors. Then we establish the path-connectedness of the space of oriented transversal triangles. Finally, we calculate the trace of the holonomy of an oriented triangle (see 2.5.1) and prove that counterclockwise-oriented transversal triangles cannot be R-parabolic nor trivial.

4.4.1. Throughout this subsection, g_1, g_2, g_3 denote positive points such that $\operatorname{ta}(g_i, g_j) > 1$ for $i \ne j$. Put

$$t_{ij} := \sqrt{\operatorname{ta}(g_i, g_j)} \text{ for } i \ne j, \qquad \varkappa := \frac{\langle g_1, g_2\rangle\langle g_2, g_3\rangle\langle g_3, g_1\rangle}{\langle g_1, g_1\rangle\langle g_2, g_2\rangle\langle g_3, g_3\rangle}, \qquad \varepsilon := \frac{\varkappa}{|\varkappa|}, \qquad G := \begin{pmatrix} 1 & t_{12} & t_{31}\overline{\varepsilon} \\ t_{12} & 1 & t_{23} \\ t_{31}\varepsilon & t_{23} & 1 \end{pmatrix},$$

$$\varepsilon_0 := \operatorname{Re}\varepsilon, \qquad \varepsilon_1 := \operatorname{Im}\varepsilon, \qquad d := \det G = 1 + 2t_{12}t_{23}t_{31}\varepsilon_0 - t_{12}^2 - t_{23}^2 - t_{31}^2, \qquad S_i := \mathbb{P}g_i^\perp.$$

We fix suitable representatives so that G is the Gram matrix of $g_1, g_2, g_3 \in V$. The numbers t_{ij}'s and ε are invariant under the action of PU on the ordered triples (g_1, g_2, g_3) and constitute a complete set of invariants. Indeed, if the triples (g_1, g_2, g_3) and (g_1', g_2', g_3') have the same Gram matrix, then there exists some $X \in U$ such that $Xg_i = g_i'$. By Sylvester's Criterion [KoM, p. 113], all the values of the t_{ij}'s and ε subject to the conditions $t_{ij} > 1$, $|\varepsilon| = 1$, and $d \le 0$ are possible. The equality $d = 0$ means that g_1, g_2, g_3 are in the same projective line.

Recall that the triangle of bisectors $\Delta(S_1, S_2, S_3)$ is *transversal* if the bisectors $\mathrm{B}\wr g_i, g_j\wr$ and $\mathrm{B}\wr g_i, g_k\wr$ are transversal along their common slice $S_i \cap \overline{\mathbb{B}}$ for all i, j, k. The triangle $\Delta(S_1, S_2, S_3)$ is said to be *counterclockwise-oriented* if the oriented angle from $\mathrm{B}[S_2, S_3]$ to $\mathrm{B}[S_2, S_1]$ at some point $p \in S_2 \cap \mathbb{B}$ does not exceed π (see Definition 2.3.5).

Criterion 4.4.2. *A transversal triangle of bisectors is counterclockwise-oriented if and only if $\varepsilon_1 < 0$. Suppose that $1 < t_{12} \le t_{23}, t_{31}$. Then $\Delta(S_1, S_2, S_3)$ is transversal if and only if $t_{12}^2\varepsilon_0^2 + t_{23}^2 + t_{31}^2 < 1 + 2t_{12}t_{23}t_{31}\varepsilon_0$.*

Proof. For the first assertion, we measure the oriented angle from $\mathrm{B}[S_2, S_3]$ to $\mathrm{B}[S_2, S_1]$ at the point $p := \pi[g_2]g_1 = g_1 - t_{12}g_2 \in S_2$. (It follows from $t_{12} > 1$ that $p \in S_2 \cap \mathbb{B}$.) By Lemma 4.3.1, this angle equals

$$\operatorname{Arg}\frac{\langle g_3, p\rangle\langle p, g_1\rangle}{\langle g_3, g_2\rangle\langle g_2, g_1\rangle} = \operatorname{Arg}\big((t_{31}\varepsilon - t_{12}t_{23})(1 - t_{12}^2)\big) = \operatorname{Arg}(t_{12}t_{23} - t_{31}\varepsilon),$$

where Arg takes values in $[0, 2\pi)$. The triangle is counterclockwise-oriented exactly when $\mathrm{Im}(t_{12}t_{23} - t_{31}\varepsilon) > 0$, that is, when $\varepsilon_1 < 0$.

By Criterion 4.3.3, the transversality of $\mathrm{B}\langle g_1, g_2 \rangle$ and $\mathrm{B}\langle g_2, g_3 \rangle$ along $S_2 \cap \overline{\mathbb{B}}$ is equivalent to the inequality

$$\left| \mathrm{Re}\, \frac{t_{31}\overline{\varepsilon}}{t_{12}t_{23}} - 1 \right| < \sqrt{1 - \frac{1}{t_{12}^2}} \cdot \sqrt{1 - \frac{1}{t_{23}^2}}.$$

This inequality is equivalent to $(t_{31}\varepsilon_0 - t_{12}t_{23})^2 < \sqrt{(t_{12}^2 - 1)(t_{23}^2 - 1)}$, that is, to

$$t_{12}^2 + t_{23}^2 + t_{31}^2\varepsilon_0^2 < 1 + 2t_{12}t_{23}t_{31}\varepsilon_0.$$

By symmetry, the other two transversalities are equivalent to the inequalities

$$t_{12}^2 + t_{23}^2\varepsilon_0^2 + t_{31}^2 < 1 + 2t_{12}t_{23}t_{31}\varepsilon_0, \qquad t_{12}^2\varepsilon_0^2 + t_{23}^2 + t_{31}^2 < 1 + 2t_{12}t_{23}t_{31}\varepsilon_0.$$

The last inequality implies the other two due to $t_{12} \leq t_{23}, t_{31}$. ∎

The path-connectedness of the space of oriented transversal triangles of bisectors is the subject of the following

Lemma 4.4.3. *The region given in $\mathbb{R}^4(e, t_1, t_2, t_3)$ by the inequalities $1 < t_1 \leq t_2, t_3$ and $t_1^2e^2 + t_2^2 + t_3^2 < 1 + 2t_1t_2t_3e \leq t_1^2 + t_2^2 + t_3^2$ is path-connected.*

Proof. The inequalities in the lemma imply $e^2 < 1$ and, hence, $t_1e < t_2t_3$. The inequality $t_1^2e^2 + t_2^2 + t_3^2 < 1 + 2t_1t_2t_3e$ can be rewritten as $(t_2t_3 - t_1e)^2 < (t_2^2 - 1)(t_3^2 - 1)$. Therefore, it is equivalent to the inequalities $t_2t_3 - \sqrt{(t_2^2 - 1)(t_3^2 - 1)} < t_1e \leq t_2t_3$. Now, from the inequality $1 \leq t_2t_3 - \sqrt{(t_2^2 - 1)(t_3^2 - 1)}$ implied by $1 < t_2t_3$, we conclude that $1 < t_1e$. In particular, $0 < e < 1$.

The inequalities

$$1 < t_1 \leq t_2, t_3, \qquad t_2t_3 - \sqrt{(t_2^2 - 1)(t_3^2 - 1)} < t_1e \leq t_2t_3, \qquad 1 + 2t_1t_2t_3e \leq t_1^2 + t_2^2 + t_3^2 \qquad \textbf{(4.4.4)}$$

are equivalent to those in the lemma. We have seen that they imply $0 < e < 1 < t_1e$.

Suppose that e, t_1, t_2, t_3 satisfy the inequalities in the lemma. Keeping the inequalities (4.4.4), we will vary the t_i's until we reach $t_1 = t_2 = t_3$ or $1 + 2t_1t_2t_3e = t_1^2 + t_2^2 + t_3^2$. Without loss of generality, we can assume that $t_2 \leq t_3$. It follows from the inequality $t_2 < t_2t_3e$ (implied by $1 < t_1e$ and $t_1 \leq t_3$) that the function $f(x) := x^2 + t_2^2 + t_3^2 - 2xt_2t_3e - 1$ is decreasing in $x \in [t_1, t_2]$. Increasing t_1 and keeping the inequalities (4.4.4) and $t_2 \leq t_3$, we can reach a point such that either $t_1 = t_2$ or $f(t_1) = 0$. The latter means that $1 + 2t_1t_2t_3e = t_1^2 + t_2^2 + t_3^2$.

In the case of $t_1 = t_2$, we can rewrite the condition $t_1 = t_2$ and the inequalities in the lemma as

$$1 < t_1 = t_2 \leq t_3, \qquad 2t_1^2(t_3e - 1) \leq t_3^2 - 1 < t_1^2(2t_3e - e^2 - 1). \qquad \textbf{(4.4.5)}$$

As we saw, these conditions imply that $0 < e < 1 < t_1e$. Hence, $0 < t_3e - 1$ and $0 < 2t_3e - e^2 - 1$. Now, we increase $t_1 = t_2$ keeping the conditions (4.4.5) and reach a point where $t_1 = t_2 = t_3$ or $2t_1^2(t_3e - 1) = t_3^2 - 1$. The latter means that $1 + 2t_1t_2t_3e = t_1^2 + t_2^2 + t_3^2$.

In the case of $t_1 = t_2 = t_3$, the inequalities (4.4.4) take the form

$$1 < t_1 = t_2 = t_3, \qquad 1 < t_1e \leq t_1^2, \qquad t_1e \leq \frac{3t_1^2 - 1}{2t_1^2}. \qquad \textbf{(4.4.6)}$$

It follows from $\dfrac{3t_1^2 - 1}{2t_1^2} < t_1^2$ (implied by $1 < t_1$) that we can increase e and keep the conditions (4.4.6)

until $t_1 e = \dfrac{3t_1^2 - 1}{2t_1^2}$. Again, we reach a point satisfying the equality $1 + 2t_1 t_2 t_3 e = t_1^2 + t_2^2 + t_3^2$.

It remains to show that the region given by the inequalities in the lemma and by the equality $1 + 2t_1 t_2 t_3 e = t_1^2 + t_2^2 + t_3^2$ is path-connected. Excluding e with the help of the equality, we can describe this region as the one given in $\mathbb{R}^3(t_1, t_2, t_3)$ by the inequalities

$$1 < t_1 \le t_2, t_3, \qquad t_1^2 + t_2^2 + t_3^2 < 1 + 2t_1 t_2 t_3. \tag{4.4.7}$$

Assuming that $t_2 \le t_3$, we obtain $t_2 < t_1 t_3$. Therefore, keeping the inequalities (4.4.7), we can increase t_2 until $t_2 = t_3$. Now, the inequalities (4.4.7) are equivalent to the conditions $1 < t_1 \le t_2 = t_3$. $\blacksquare$

4.4.8. Trace of holonomy. Let us recall the definition of the holonomy φ of $\Delta(S_1, S_2, S_3)$. Put

$$m_1 := \frac{g_1 + g_2}{\sqrt{2t_{12} + 2}}, \qquad m_2 := \frac{g_2 + g_3}{\sqrt{2t_{23} + 2}}, \qquad m_3 := \frac{\varepsilon g_1 + g_3}{\sqrt{2t_{31} + 2}}. \tag{4.4.9}$$

By Lemma 4.1.22, $R_i := R(m_i)$ is the reflection (see (2.1.3) for the definition) in the middle slice of the segment $\mathrm{B}[S_i, S_{i+1}]$ and $\langle m_i, m_i \rangle = 1$ (the indices are modulo 3). Define $\varphi := R_3 R_2 R_1 \in \mathrm{SU}$. It is easy to see that $R_1 g_1 = g_2$, $R_2 g_2 = g_3$, and $R_3 g_3 = \varepsilon g_1$. Hence,

$$\varphi g_1 = \varepsilon g_1. \tag{4.4.10}$$

In particular, $g_1^\perp$ is stable under φ. Therefore, the isometry φ induces an isometry $\psi \in \mathrm{SU}(g_1^\perp)$ of the complex geodesic $\mathbb{P}g_1^\perp$.

Lemma 4.4.11. $|\operatorname{tr}\psi| = \sqrt{2(1 + \varepsilon_0)}\Big(1 - \dfrac{d}{(t_{12} + 1)(t_{23} + 1)(t_{31} + 1)}\Big).$

Proof. Since $\varphi \in \mathrm{SU}$ and $\varphi g_1 = \varepsilon g_1$, it suffices to show that

$$\operatorname{tr}\varphi = \varepsilon - (1 + \overline{\varepsilon})\Big(1 - \frac{d}{(t_{12} + 1)(t_{23} + 1)(t_{31} + 1)}\Big). \tag{4.4.12}$$

Indeed, (4.4.12) implies that $\operatorname{tr}\psi = \pm(\sqrt{\varepsilon} + \overline{\sqrt{\varepsilon}})\Big(1 - \dfrac{d}{(t_{12} + 1)(t_{23} + 1)(t_{31} + 1)}\Big)$ because $\det\varphi = 1$ and $\varphi g_1 = \varepsilon g_1$.

Define the linear maps $\varphi_i \in \operatorname{Lin}_{\mathbb{C}}(V, V)$ by the rule $\varphi_i x := 2\langle x, m_i \rangle m_i$ and denote $g_{ij} := \langle m_i, m_j \rangle$, $i, j = 1, 2, 3$. Obviously, $R_i = \varphi_i - 1$ and

$$\varphi_j \varphi_i x = 4\langle x, m_i \rangle g_{ij} m_j, \qquad \varphi_k \varphi_j \varphi_i x = 8\langle x, m_i \rangle g_{ij} g_{jk} m_k.$$

Considering the orthogonal decompositions $V = \mathbb{C} m_i \oplus m_i^\perp$, we obtain

$$\operatorname{tr}\varphi_i = 2, \qquad \operatorname{tr}(\varphi_j \varphi_i) = 4g_{ij} g_{ji}, \qquad \operatorname{tr}(\varphi_k \varphi_j \varphi_i) = 8g_{ij} g_{jk} g_{ki}.$$

It follows from $\varphi = R_3 R_2 R_1 = \varphi_3 \varphi_2 \varphi_1 - \varphi_3 \varphi_2 - \varphi_3 \varphi_1 - \varphi_2 \varphi_1 + \varphi_3 + \varphi_2 + \varphi_1 - 1$ that

$$\operatorname{tr}\varphi = 8g_{12} g_{23} g_{31} - 4g_{23} g_{32} - 4g_{13} g_{31} - 4g_{12} g_{21} + 3 \tag{4.4.13}$$

(compare with [Pra]). From (4.4.9), we obtain

$$g_{12} = \frac{1 + t_{12} + t_{23} + t_{31}\overline{\varepsilon}}{2\sqrt{(t_{12}+1)(t_{23}+1)}}, \qquad g_{23} = \frac{1 + t_{23} + t_{31} + t_{12}\overline{\varepsilon}}{2\sqrt{(t_{23}+1)(t_{31}+1)}}, \qquad g_{31} = \frac{\varepsilon + t_{31}\varepsilon + t_{12}\varepsilon + t_{23}}{2\sqrt{(t_{31}+1)(t_{12}+1)}},$$

$$4g_{23}g_{32} = 4\operatorname{ta}(m_2, m_3) = \frac{(1 + t_{23} + t_{31})^2 + t_{12}^2 + 2t_{12}(1 + t_{23} + t_{31})\varepsilon_0}{(t_{23}+1)(t_{31}+1)} =$$

$$= 2 + \frac{2t_{12}(1 + t_{23} + t_{31} + t_{23}t_{31})\varepsilon_0 - d}{(t_{23}+1)(t_{31}+1)} = 2 + 2t_{12}\varepsilon_0 - \frac{d(t_{12}+1)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}.$$

By symmetry,

$$4g_{13}g_{31} = 2 + 2t_{23}\varepsilon_0 - \frac{d(t_{23}+1)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}, \qquad 4g_{12}g_{21} = 2 + 2t_{31}\varepsilon_0 - \frac{d(t_{31}+1)}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}.$$

It is possible to show with a straightforward calculation that

$$8g_{12}g_{23}g_{31}(t_{12}+1)(t_{23}+1)(t_{31}+1) =$$

$$= (t_{12}+1)(t_{23}+1)(t_{31}+1)\big(2(t_{12}+t_{23}+t_{31})\varepsilon_0 + 2 + \varepsilon - \overline{\varepsilon}\big) - d(t_{12}+t_{23}+t_{31}+2-\overline{\varepsilon}).$$

In other words,

$$8g_{12}g_{23}g_{31} = 2(t_{12}+t_{23}+t_{31})\varepsilon_0 + 2 + \varepsilon - \overline{\varepsilon} - \frac{d(t_{12}+t_{23}+t_{31}+2-\overline{\varepsilon})}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}.$$

Now, from (4.4.13), we infer (4.4.12).   ∎

4.4.14. Parabolic holonomy. Remind that a triangle $\Delta(S_1, S_2, S_3)$ is *parabolic* or *trivial* if the isometry ψ (see 4.4.8) is parabolic or the identity. An oriented parabolic triangle is L-*parabolic* if ψ moves the points in $S_1 \cap \partial_\infty \mathbb{B}$ (different from the fixed one) in the clockwise direction.

Lemma 4.4.15. *Let $\Delta(S_1, S_2, S_3)$ be a counterclockwise-oriented transversal triangle. Suppose that $\Delta(S_1, S_2, S_3)$ is parabolic or trivial. Then $\Delta(S_1, S_2, S_3)$ is L-parabolic.*

Proof. Denote $p := \pi[g_1]g_2 \in S_1$. It follows from $t_{12} > 1$ that $p \in S_1 \cap \mathbb{B}$. Let $q \in S_1 \cap \partial_\infty \mathbb{B}$ denote a fixed point of φ. We assume that $\langle p, q \rangle = 1$ and define $p' := \langle p, p \rangle q - p \in p^\perp$. Since $\langle p', p' \rangle = -\langle p, p \rangle$, every point in $S_1 \cap \overline{\mathbb{B}}$ has the form

$$p(z) := zp' + p, \qquad z \in \mathbb{C}, \qquad |z| \le 1,$$

in the orthogonal basis $p, p' \in g_1^\perp$. Clearly, $p(1) \sim q$, where $\sim$ stands for $\mathbb{C}^*$-proportionality.

Being φ parabolic or trivial over $S_1 = L\wr p, q\wr$ with the fixed point q, we have

$$\varphi p = up + uivq, \qquad \varphi q = uq \tag{4.4.16}$$

for suitable $u, v \in \mathbb{C}$, $u \ne 0$. This implies that

$$\operatorname{tr}\varphi = 2u + \varepsilon$$

because $\varphi g_1 = \varepsilon g_1$ by (4.4.10). From $\varphi \in \mathrm{SU}$, $\langle p, q \rangle = 1$, and $\langle q, q \rangle = 0$, we conclude that

$$1 = \langle p, q \rangle = \langle \varphi p, \varphi q \rangle = |u|^2, \qquad \langle p, p \rangle = \langle \varphi p, \varphi p \rangle = |u|^2\big(\langle p, p \rangle - i\overline{v} + iv\big) = \langle p, p \rangle - 2\operatorname{Im}v.$$

Hence, $v \in \mathbb{R}$.

Let us understand how does φ act on S_1 in terms of z :

$$\varphi\big(p(z)\big) = z\varphi\big(\langle p, p\rangle q - p\big) + \varphi p \sim z\langle p, p\rangle q + (1-z)(p + ivq) =$$

$$= \frac{z\langle p, p\rangle + (1-z)iv}{\langle p, p\rangle}\big(\langle p, p\rangle q - p\big) + \frac{\langle p, p\rangle + (1-z)iv}{\langle p, p\rangle}p \sim p(z'),$$

where

$$z' := \frac{z\langle p, p\rangle + (1-z)iv}{\langle p, p\rangle + (1-z)iv}.$$

In particular, $\varphi\big(p(-1)\big) \sim p(z_0)$ with $z_0 := \dfrac{2iv - \langle p, p\rangle}{2iv + \langle p, p\rangle}$ and $\operatorname{Im} z_0 = \dfrac{4v\langle p, p\rangle}{\big|2iv + \langle p, p\rangle\big|^2}$. We can see that φ moves $p(-1)$ in the clockwise direction exactly when $\operatorname{Im} z_0 > 0$, i.e., if and only if $v < 0$. In other words, the triangle is L-parabolic if and only if $v < 0$. It is easy to see from (4.4.16) that the triangle is trivial if and only if $v = 0$.

Thus, we have to show that $v < 0$.

It follows from $\langle p, q\rangle = 1$ and (4.4.16) that $\langle \varphi p, p\rangle = u\big(\langle p, p\rangle + iv\big)$, i.e., $v = \operatorname{Im}\big(u^{-1}\langle \varphi p, p\rangle\big)$. By (4.4.12),

$$\operatorname{tr}\varphi = 2u + \varepsilon = \varepsilon - (1 + \overline{\varepsilon})\Big(1 - \frac{d}{(t_{12} + 1)(t_{23} + 1)(t_{31} + 1)}\Big)$$

with $d \leq 0$. Consequently, v and $\operatorname{Im}\big(-(1+\varepsilon)\langle \varphi p, p\rangle\big)$ have the same signs.

Let us calculate $\langle \varphi p, p\rangle$ in terms of the Gram matrix G. We only need to find φg_2 since $p = g_2 - t_{12}g_1$ and $\varphi g_1 = \varepsilon g_1$ by (4.4.10). Using (4.4.9) and (2.1.3), we can see that

$$R_1 x = \langle x, g_1 + g_2\rangle\frac{g_1 + g_2}{t_{12} + 1} - x, \qquad R_2 x = \langle x, g_2 + g_3\rangle\frac{g_2 + g_3}{t_{23} + 1} - x, \qquad R_3 x = \langle x, \varepsilon g_1 + g_3\rangle\frac{\varepsilon g_1 + g_3}{t_{31} + 1} - x$$

for all $x \in V$. So, $R_1 g_2 = g_1$, $R_2 g_1 = \dfrac{t_{12} + t_{31}\overline{\varepsilon}}{t_{23} + 1}(g_2 + g_3) - g_1$, and

$$\varphi g_2 = \Big\langle \frac{t_{12} + t_{31}\overline{\varepsilon}}{t_{23} + 1}(g_2 + g_3) - g_1, \varepsilon g_1 + g_3\Big\rangle\frac{\varepsilon g_1 + g_3}{t_{31} + 1} - \frac{t_{12} + t_{31}\overline{\varepsilon}}{t_{23} + 1}(g_2 + g_3) + g_1 =$$

$$= \Big(\frac{t_{12} + t_{31}\overline{\varepsilon}}{t_{23} + 1}(t_{12}\overline{\varepsilon} + t_{31} + t_{23} + 1) - \overline{\varepsilon} - t_{31}\overline{\varepsilon}\Big)\frac{\varepsilon g_1 + g_3}{t_{31} + 1} - \frac{t_{12} + t_{31}\overline{\varepsilon}}{t_{23} + 1}(g_2 + g_3) + g_1 =$$

$$= \frac{(t_{12} + t_{31}\overline{\varepsilon})(t_{12}\overline{\varepsilon} + t_{31}) + (t_{23} + 1)(t_{12} - \overline{\varepsilon})}{(t_{23} + 1)(t_{31} + 1)}(\varepsilon g_1 + g_3) - \frac{t_{12} + t_{31}\overline{\varepsilon}}{t_{23} + 1}(g_2 + g_3) + g_1.$$

Therefore, taking into account that $\langle g_1, g_2 - t_{12}g_1\rangle = 0$, we obtain

$$(t_{23} + 1)(t_{31} + 1)\langle \varphi p, p\rangle = (t_{23} + 1)(t_{31} + 1)\langle \varphi g_2, g_2 - t_{12}g_1\rangle =$$

$$= (t_{23} + 1)(t_{31} + 1)\Big\langle \frac{(t_{12} + t_{31}\overline{\varepsilon})(t_{12}\overline{\varepsilon} + t_{31}) + (t_{23} + 1)(t_{12} - \overline{\varepsilon})}{(t_{23} + 1)(t_{31} + 1)}g_3 - \frac{t_{12} + t_{31}\overline{\varepsilon}}{t_{23} + 1}(g_2 + g_3), g_2 - t_{12}g_1\Big\rangle =$$

$$= (t_{12} + t_{31}\overline{\varepsilon})(t_{12}\overline{\varepsilon} + t_{31})(t_{23} - t_{12}t_{31}\varepsilon) + (t_{23} + 1)(t_{12} - \overline{\varepsilon})(t_{23} - t_{12}t_{31}\varepsilon) -$$

$$- (t_{12} + t_{31}\overline{\varepsilon})(t_{31} + 1)(1 - t_{12}^2 + t_{23} - t_{12}t_{31}\varepsilon) =$$

$$= t_{12}t_{23}t_{31}\overline{\varepsilon}^2 + (t_{12}^2 t_{23} + t_{12}^2 t_{31} - t_{23}^2 - t_{23}t_{31} - t_{31}^2 - t_{23} - t_{31})\overline{\varepsilon} + t_{12}^3 + t_{12}t_{23}^2 + t_{12}t_{23}t_{31} + t_{12}t_{31}^2 - t_{12} - t_{12}^2 t_{23}t_{31}\varepsilon.$$

Hence,

$$\mathrm{Im}\left(-(t_{23}+1)(t_{31}+1)(1+\varepsilon)\langle\varphi p, p\rangle\right) = t_{12}^2 t_{23}t_{31}\,\mathrm{Im}\,\varepsilon^2 - t_{12}t_{23}t_{31}\,\mathrm{Im}\,\overline{\varepsilon}^2 -$$

$$-(t_{12}^2 t_{23} + t_{12}^2 t_{31} + t_{12}t_{23}t_{31} - t_{23}^2 - t_{23}t_{31} - t_{31}^2 - t_{23} - t_{31})\,\mathrm{Im}\,\overline{\varepsilon} -$$

$$-(t_{12}^3 + t_{12}t_{23}^2 + t_{12}t_{23}t_{31} + t_{12}t_{31}^2 - t_{12} - t_{12}^2 t_{23}t_{31})\,\mathrm{Im}\,\varepsilon =$$

$$= (t_{12}+1)\varepsilon_1\left(1 + 2t_{12}t_{23}t_{31}\varepsilon_0 - t_{12}^2 - t_{23}^2 - t_{31}^2 + (t_{12}-1)(t_{23}+1)(t_{31}+1)\right) =$$

$$= (t_{12}+1)\varepsilon_1\left(d + (t_{12}-1)(t_{23}+1)(t_{31}+1)\right).$$

By Lemma 4.4.11,

$$\sqrt{2(1+\varepsilon_0)}\left(1 - \frac{d}{(t_{12}+1)(t_{23}+1)(t_{31}+1)}\right) = 2$$

because the triangle is parabolic or trivial. So,

$$d = \left(1 - \frac{2}{\sqrt{2(1+\varepsilon_0)}}\right)(t_{12}+1)(t_{23}+1)(t_{31}+1).$$

We can see that the sign of v is that of $\varepsilon_1\left(t_{12} - \dfrac{t_{12}+1}{\sqrt{2(1+\varepsilon_0)}}\right)$. By Criterion 4.4.2, $\varepsilon_1 < 0$ since the triangle is counterclockwise-oriented. It remains to prove that $0 < t_{12} - \dfrac{t_{12}+1}{\sqrt{2(1+\varepsilon_0)}}$, i.e., that $2 < (t_{12}-1)^2 + 2t_{12}^2\varepsilon_0$. It suffices to show that $1 < t_{12}\varepsilon_0$:

The inequality $1 \le t_{23}t_{31} - \sqrt{(t_{23}^2-1)(t_{31}^2-1)}$ follows from $1 < t_{23}t_{31}$. By Criterion 4.4.2, we have $t_{12}^2\varepsilon_0^2 + t_{23}^2 + t_{31}^2 < 1 + 2t_{12}t_{23}t_{31}\varepsilon_0$ or, equivalently, $(t_{23}t_{31} - t_{12}\varepsilon_0)^2 < (t_{23}^2-1)(t_{31}^2-1)$. The inequality $t_{12}\varepsilon_0 \le t_{23}t_{31}$ provides $t_{23}t_{31} - \sqrt{(t_{21}^2-1)(t_{31}^2-1)} < t_{12}\varepsilon_0$. ∎

4.4.17. $\mathbb{C}$-plane triangle. Assume $d = 0$. This means that the triangle of bisectors $\Delta(S_1, S_2, S_3)$ is built over a usual geodesic triangle $\Delta(p_1, p_2, p_3)$ situated in a complex geodesic C of signature $+-$ which is actually the common complex spine of the bisectors. In other words, $\mathrm{B}[S_i, S_j] = \mathrm{B}[p_i, p_j]$, where p_i is orthogonal to g_i in C (see Remark 4.1.10).

The triangle $\Delta(S_1, S_2, S_3)$ is always transversal. This follows from [Mos, p. 186, Lemma 2.3.4] (Mostow shows that the angle between cospinal bisectors is constant) or from Criterion 4.4.2 : The inequality $t_{12}^2\varepsilon_0^2 + t_{23}^2 + t_{31}^2 < 1 + 2t_{12}t_{23}t_{31}\varepsilon_0$ follows from $1 + 2t_{12}t_{23}t_{31}\varepsilon_0 = t_{12}^2 + t_{23}^2 + t_{31}^2$ and $\varepsilon_0^2 < 1$. (Note that $\varepsilon_0^2 = 1$ would imply that the points g_1, g_2, g_3 belong to a same geodesic by Lemma 4.1.13, providing that p_1, p_2, p_3 belong to the same geodesic by Remark 4.1.10.) Obviously, $\Delta(S_1, S_2, S_3)$ is counterclockwise-oriented exactly when $\Delta(p_1, p_2, p_3)$ is counterclockwise-oriented in C.

Lemma 4.4.18. *Suppose that $\Delta(S_1, S_2, S_3)$ is counterclockwise-oriented and $\mathbb{C}$-plane. Then φ (see 4.4.8 for the definition) restricted to S_1 is a rotation about p_1 by the angle $-2\,\mathrm{Area}\,\Delta(p_1, p_2, p_3)$.*

Proof. Let $f \notin \overline{\mathbb{B}}$ denote the polar point to the complex geodesic C containing $\Delta(p_1, p_2, p_3)$. Obviously, $\varphi f = -f$. By (4.4.10), $\varphi g_1 = \varepsilon g_1$. Taking into account that the points f, p_1, g_1 form an orthogonal basis and that $\varphi \in \mathrm{SU}$, we obtain $\varphi p_1 = -\overline{\varepsilon}p_1$. Assuming that $\langle f, f\rangle = 1$ and $\langle p_1, p_1\rangle = -1$, we write every point in $S_1 \cap \overline{\mathbb{B}}$ in the form $zf + p_1$, $|z| \le 1$. In terms of z, the isometry φ acts on $S_1 \cap \overline{\mathbb{B}}$ as the multiplication by ε. So, the restriction of φ to $S_1 \cap \overline{\mathbb{B}}$ is the rotation about p_1 by the angle $\arg\varepsilon \in [-\pi, \pi]$. It remains to show that $2\,\mathrm{Area}\,\Delta(p_1, p_2, p_3) = -\arg\varepsilon$.

We apply to the representatives $\pi[g_1]g_2, \pi[g_2]g_1, \pi[g_3]g_2$ of p_1, p_2, p_3 the known formula

$$\mathrm{Area}\,\Delta(p_1, p_2, p_3) = \tfrac{1}{2}\arg\left(-\langle p_1, p_2\rangle\langle p_2, p_3\rangle\langle p_3, p_1\rangle\right)$$

for the area of the oriented triangle $\Delta(p_1, p_2, p_3)$ (see [AGr1] or [Gol, p. 25]; by Corollary 4.1.18, the metric we use differs by the factor of 4 from the one in [Gol]) and obtain

$$2\operatorname{Area}\Delta(p_1,p_2,p_3) = \arg\big(-\langle g_2 - t_{12}g_1, g_1 - t_{12}g_2\rangle\langle g_1 - t_{12}g_2, g_2 - t_{23}g_3\rangle\langle g_2 - t_{23}g_3, g_2 - t_{12}g_1\rangle\big) =$$

$$= \arg\big(-t_{12}(t_{12}^2 - 1)t_{23}(t_{12}t_{23} - t_{31}\overline{\varepsilon})(1 - t_{12}^2 - t_{23}^2 + t_{12}t_{23}t_{31}\varepsilon)\big) =$$

$$= \arg\big((t_{12}t_{23} - t_{31}\overline{\varepsilon})(t_{12}^2 + t_{23}^2 - t_{12}t_{23}t_{31}\varepsilon - 1)\big).$$

It follows from $1 + 2t_{12}t_{23}t_{31}\varepsilon_0 = t_{12}^2 + t_{23}^2 + t_{31}^2$ that

$$t_{12}^2 + t_{23}^2 - t_{12}t_{23}t_{31}\varepsilon - 1 = 2t_{12}t_{23}t_{31}\varepsilon_0 - t_{12}t_{23}t_{31}\varepsilon - t_{31}^2 = t_{31}(t_{12}t_{23}\overline{\varepsilon} - t_{31}) = t_{31}(t_{12}t_{23} - t_{31}\varepsilon)\overline{\varepsilon},$$

which implies the result.   ∎

4.5. Kähler potential

With respect to the hermitian form (4.1.2), we define on $\mathbb{P} \setminus \partial_\infty\mathbb{B}$ a *Kähler form* ω by the rule $\omega(v_p, w_p) := \operatorname{Im}\langle v_p, w_p\rangle$, where $v_p, w_p \in \mathrm{T}_p\mathbb{P}$ (see (4.1.3) for the definition of v_p). In this subsection, we obtain a potential P_c for ω that depends on the choice of a *base* point c and study how this potential changes when we alter the base point.

4.5.1. Let $c \in \mathbb{P}$, let $p \in \mathbb{P} \setminus (\mathbb{P}c^\perp \cup \partial_\infty\mathbb{B})$, and let $v_p \in \mathrm{T}_p\mathbb{P}$. Define

$$P_c(v_p) := -\operatorname{Im}\Big(\frac{\langle p, p\rangle\langle v, c\rangle}{2\langle p, c\rangle}\Big).$$

Note that $P_c(v_p)$ does not depend on the choice of representatives $c, p, v \in V$ that provide the same v_p. Obviously, $P_c(v_p)$ depends smoothly on $v_p \in \mathrm{T}_p\mathbb{P}$, i.e., it defines a 1-form on $\mathbb{P} \setminus (\mathbb{P}c^\perp \cup \partial_\infty\mathbb{B})$.

Let $c_1, c_2 \in \mathbb{P}$ be nonorthogonal. Define the multi-valued function

$$f_{c_1,c_2}(p) := \frac{1}{2}\operatorname{Arg}\Big(\frac{\langle c_1, p\rangle\langle p, c_2\rangle}{\langle c_1, c_2\rangle}\Big),$$

where $p \in \mathbb{P} \setminus (\mathbb{P}c_1^\perp \cup \mathbb{P}c_2^\perp)$. Note that $f_{c_1,c_2}(p)$ does not depend on the choice of representatives $c_1, c_2, p \in V$.

Proposition 4.5.2. *For nonorthogonal* $c_1, c_2 \in \mathbb{P}$, *we have* $P_{c_1} - P_{c_2} = df_{c_1,c_2}$ *on* $\mathbb{P}\setminus(\partial_\infty\mathbb{B}\cup\mathbb{P}c_1^\perp\cup\mathbb{P}c_2^\perp)$.

Proof. Choose representatives $c_1, c_2 \in V$ such that $\langle c_1, c_2\rangle \in \mathbb{R}$. Take $p \in \mathbb{P} \setminus (\partial_\infty\mathbb{B} \cup \mathbb{P}c_1^\perp \cup \mathbb{P}c_2^\perp)$ and $v_p \in \mathrm{T}_p\mathbb{P}$. Then

$$df_{c_1,c_2}(v_p) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f_{c_1,c_2}\big(p + \varepsilon\langle p, p\rangle v\big) = \frac{1}{2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\operatorname{Arg}\Big(\langle c_1, p + \varepsilon\langle p, p\rangle v\rangle\langle p + \varepsilon\langle p, p\rangle v, c_2\rangle\Big) =$$

$$= \frac{1}{2}\operatorname{Im}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\ln\langle c_1, p + \varepsilon\langle p, p\rangle v\rangle + \ln\langle p + \varepsilon\langle p, p\rangle v, c_2\rangle\Big) =$$

$$= \frac{1}{2}\operatorname{Im}\Big(\frac{\langle p, p\rangle\langle c_1, v\rangle}{\langle c_1, p\rangle} + \frac{\langle p, p\rangle\langle v, c_2\rangle}{\langle p, c_2\rangle}\Big) = \big(P_{c_1} - P_{c_2}\big)(v_p).\qquad\blacksquare$$

4.5.3. In order to show that P_c is a potential for the form ω, we introduce an auxiliary vector field. Let $p \in \mathbb{P} \setminus \partial_\infty\mathbb{B}$ and let $v_p \in \mathrm{T}_p\mathbb{P}$. The smooth vector field

$$\operatorname{Tn}(v_p)(x) := \langle -, x\rangle\frac{\langle p, p\rangle\pi[x]v}{\langle p, x\rangle}$$

is defined on $\mathbb{P} \setminus (\mathbb{P}p^\perp \cup \partial_\infty \mathbb{B})$. This field extends the vector v_p, i.e., $\mathrm{Tn}(v_p)(p) = v_p$. Note that $\mathrm{Tn}(v_p)(x)$ does not depend on the choice of representatives $p, v, x \in V$ that provide the same v_p.

Lemma 4.5.4. *Let $p \in \mathbb{P} \setminus \partial_\infty \mathbb{B}$ and let $v_p, w_p \in \mathrm{T}_p \mathbb{P}$. Then $\big[\mathrm{Tn}(v_p), \mathrm{Tn}(w_p)\big](p) = 0$.*

Proof. Denote by $\hat{f}$ the lift to V of a C^2-function f defined in a neighbourhood of p. So, $\hat{f}(ux) = \hat{f}(x)$ for $0 \neq u \in \mathbb{C}$. By definition,

$$\mathrm{Tn}(w_p)f(x) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\Big(x + \varepsilon \langle x, x \rangle \frac{\langle p, p \rangle \pi[x]w}{\langle p, x \rangle}\Big).$$

Therefore,

$$\mathrm{Tn}(v_p)\big(\mathrm{Tn}(w_p)f\big)(p) =$$

$$= \frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\Big(p + \delta\langle p, p \rangle v + \varepsilon\langle p + \delta\langle p, p \rangle v, p + \delta\langle p, p \rangle v\rangle \frac{\langle p, p \rangle \pi\big[p + \delta\langle p, p \rangle v\big]w}{\langle p, p + \delta\langle p, p \rangle v\rangle}\Big)\Big) =$$

$$= \frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\Big(p + \delta\langle p, p \rangle v + \varepsilon\big(\langle p, p \rangle + \delta^2\langle p, p \rangle^2 \langle v, v \rangle\big)w - \varepsilon\delta\langle p, p \rangle\langle w, v \rangle\big(p + \delta\langle p, p \rangle v\big)\Big)\Big) =$$

$$= \frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\Big(\big(1 - \varepsilon\delta\langle p, p \rangle\langle w, v \rangle\big)\big(p + \delta\langle p, p \rangle v\big) + \varepsilon\big(\langle p, p \rangle + \delta^2\langle p, p \rangle^2 \langle v, v \rangle\big)w\Big)\Big) =$$

$$\frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\Big(p + \delta\langle p, p \rangle v + \varepsilon\langle p, p \rangle \frac{1 + \delta^2\langle p, p \rangle\langle v, v \rangle}{1 - \varepsilon\delta\langle p, p \rangle\langle w, v \rangle} w\Big)\Big) = \frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}\big(p + \delta\langle p, p \rangle v + \varepsilon\langle p, p \rangle w\big)\Big).$$

By symmetry,

$$\mathrm{Tn}(w_p)\big(\mathrm{Tn}(v_p)f\big)(p) = \frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat{f}(p + \delta\langle p, p \rangle w + \varepsilon\langle p, p \rangle v)\Big). \qquad \blacksquare$$

Lemma 4.5.5. *Let $c \in \mathbb{P}$, let $p \in \mathbb{P} \setminus (\mathbb{P}c^\perp \cup \partial_\infty \mathbb{B})$, and let $v_p, w_p \in \mathrm{T}_p \mathbb{P}$. Then*

$$v_p P_c\big(\mathrm{Tn}(w_p)\big) = \frac{\langle p, p \rangle}{2} \, \mathrm{Im}\, \Big(\langle w, v \rangle + \frac{\langle p, p \rangle\langle v, c \rangle\langle w, c \rangle}{\langle p, c \rangle^2}\Big).$$

Proof. By definition,

$$P_c\big(\mathrm{Tn}(w_p)\big)(x) = -\,\mathrm{Im}\, \frac{\langle x, x \rangle\langle p, p \rangle\big\langle \pi[x]w, c \big\rangle}{2\langle x, c \rangle\langle p, x \rangle} =$$

$$= -\,\mathrm{Im}\, \frac{\langle x, x \rangle\langle p, p \rangle\langle w, c \rangle - \langle p, p \rangle\langle w, x \rangle\langle x, c \rangle}{2\langle x, c \rangle\langle p, x \rangle} = \frac{\langle p, p \rangle}{2} \, \mathrm{Im}\, \Big(\frac{\langle w, x \rangle}{\langle p, x \rangle} - \frac{\langle x, x \rangle\langle w, c \rangle}{\langle p, x \rangle\langle x, c \rangle}\Big).$$

Therefore,

$$v_p P_c\big(\mathrm{Tn}(w_p)\big) = \frac{\langle p, p \rangle}{2} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{Im}\, \Big(\frac{\langle w, p + \varepsilon\langle p, p \rangle v\rangle}{\langle p, p + \varepsilon\langle p, p \rangle v\rangle} - \frac{\langle p + \varepsilon\langle p, p \rangle v, p + \varepsilon\langle p, p \rangle v\rangle\langle w, c \rangle}{\langle p, p + \varepsilon\langle p, p \rangle v\rangle\langle p + \varepsilon\langle p, p \rangle v, c\rangle}\Big) =$$

$$= \frac{\langle p, p \rangle}{2} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{Im}\, \Big(\varepsilon\langle w, v \rangle - \frac{1 + \varepsilon^2\langle p, p \rangle\langle v, v \rangle}{\langle p, c \rangle + \varepsilon\langle p, p \rangle\langle v, c \rangle}\langle w, c \rangle\Big) = \frac{\langle p, p \rangle}{2} \, \mathrm{Im}\, \Big(\langle w, v \rangle + \frac{\langle p, p \rangle\langle v, c \rangle\langle w, c \rangle}{\langle p, c \rangle^2}\Big). \qquad \blacksquare$$

Proposition 4.5.6. *For every $c \in \mathbb{P}$, we have $dP_c = \omega$ on $\mathbb{P} \setminus (\partial_\infty \mathbb{B} \cup \mathbb{P}c^\perp)$.*

Proof. Let $p \in \mathbb{P} \setminus (\partial_\infty \mathbb{B} \cup \mathbb{P}c^\perp)$ and let $v_p, w_p \in \mathrm{T}_p \mathbb{P}$. Take $X := \mathrm{Tn}(v_p)$ and $Y := \mathrm{Tn}(w_p)$. By Lemma 4.5.4, $[X, Y](p) = 0$. Hence, $P_c([X, Y])(p) = 0$. By the Maurer-Cartan identity,

$$dP_c(X, Y)(p) = X\big(P_c(Y)\big)(p) - Y\big(P_c(X)\big)(p) - P_c\big([X, Y]\big)(p).$$

It follows from Lemma 4.5.5 that

$$dP_c(v_p, w_p) = \frac{\langle p, p \rangle}{2} \mathrm{Im}\left(\langle w, v \rangle + \frac{\langle p, p \rangle \langle v, c \rangle \langle w, c \rangle}{\langle p, c \rangle^2} \right) - \frac{\langle p, p \rangle}{2} \mathrm{Im}\left(\langle v, w \rangle + \frac{\langle p, p \rangle \langle w, c \rangle \langle v, c \rangle}{\langle p, c \rangle^2} \right) =$$

$$= \frac{1}{2} \mathrm{Im}\big(\langle p, p \rangle \langle w, v \rangle - \langle p, p \rangle \langle v, w \rangle \big) = \mathrm{Im}\langle v_p, w_p \rangle. \qquad \blacksquare$$

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